SYMPLECTIC RUNGE–KUTTA SEMIDISCRETIZATION FOR STOCHASTIC SCHRÖDINGER EQUATION[*]

CHUCHU CHEN[†] AND JIALIN HONG[†]

Abstract. Based on a variational principle with a stochastic forcing, we indicate that the stochastic Schrödinger equation in the Stratonovich sense is an infinite-dimensional stochastic Hamiltonian system, whose phase flow preserves symplecticity. We propose a general class of stochastic symplectic Runge–Kutta methods in the temporal direction to the stochastic Schrödinger equation in the Stratonovich sense and show that the methods preserve the charge conservation law. We present a convergence theorem on the relationship between the mean-square convergence order of a semi-discrete method and its local accuracy order. Taking the stochastic midpoint scheme as an example of stochastic symplectic Runge–Kutta methods in the temporal direction, based on the theorem we show that the mean-square convergence order of the semidiscrete scheme is 1 under appropriate assumptions.

Key words. stochastic Schrödinger equation, infinite-dimensional stochastic Hamiltonian system, symplectic structure, symplectic Runge–Kutta method, semidiscretization, mean-square convergence order

AMS subject classifications. 65C20, 65C30, 65C50

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1. Introduction. The Schrödinger equation, as one of the basic models for wave propagation, plays an essential role in various fields such as optics, quantum physics, optical fiber communications, and plasma physics (see [24] and references therein). Recently, stochastic perturbations of this equation have been investigated [3, 22, 4, 11], where the perturbations may be due to the inhomogeneous media or the thermal fluctuations, etc. More recently, attention has been paid to the study of the stochastic Schrödinger equation from both theoretical and numerical views; see [6, 7, 8, 16, 17]. The local and global existence of solution in space $\mathbb{H}^1(\mathbb{R}^n)$ for the stochastic Schrödinger equation are investigated in [8]. There are some conservation laws for the deterministic Schrödinger equation, for example, charge conservation law and energy conservation law. The evolution of these invariant quantities in the case of the stochastic Schrödinger equation also are considered in [8]. It is well known that the deterministic Schrödinger equation is an infinite-dimensional Hamiltonian system, that is to say, its phase flow preserves symplecticity; see [12, 14] for details. Jiang, Wang, and Hong [16] establish the theory about stochastic multisymplectic conservation law for stochastic Hamiltonian partial differential equations; the stochastic Schrödinger equation as a concrete example is investigated for this property. As far as we know, there has been no work concerning the infinite-dimensional stochastic Hamiltonian system, its stochastic symplecticity, and symplectic semidiscretization up to now. In this paper we present the general formula of an infinite-dimensional

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[†]State Key Laboratory of Scientific and Engineering Computing, Institute of Computational Mathematics and Scientific/Engineering Computing, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, People's Republic of China (chenchuchu@lsec.cc.ac.cn, hjl@lsec.cc.ac.cn).

stochastic Hamiltonian system via a variational principle with a stochastic forcing and investigate the stochastic symplecticity of the stochastic Schrödinger equation in the Stratonovich sense, which is shown to be an infinite-dimensional stochastic Hamiltonian system.

For the numerical approximations of the stochastic Schrödinger equation, a semidiscrete scheme for the stochastic nonlinear Schrödinger equation in the Stratonovich sense is proposed in [6]. The authors also obtain the convergence of the discrete solution in various topologies. For the stochastic nonlinear Schrödinger equation in the Itô sense with power nonlinearity, [7] analyzes the error of a semidiscrete scheme and proves that the numerical scheme has strong order $\frac{1}{2}$ in general and order 1 if the noise is additive. Using the integral representation idea, [17] proposes splitting schemes to the stochastic nonlinear Schrödinger equation in the Stratonovich sense and proves the first order convergence in non-global Lipschitz case. It is important to design a numerical scheme which preserves the properties of the original problem as much as possible. For a Hamiltonian system, symplectic methods are shown to be superior to nonsymplectic ones especially in long time computation, owing to their preservation of the qualitative property, the symplecticity of the underlying continuous differential equation system; see [19, 25] and references therein. Since the stochastic Schrödinger equation in the Stratonovich sense is an infinite-dimensional stochastic Hamiltonian system, we investigate numerical methods to preserve symplecticity.

In the present paper, we present a general class of Runge–Kutta methods in the temporal direction for the stochastic Schrödinger equation first and then obtain the symplectic conditions for Runge–Kutta methods in the temporal direction. Under the symplectic conditions of Runge–Kutta methods, we show that they preserve the charge conservation law. Adopting the idea of the work [19], which establishes the mean-square order of convergence of a method resting on properties of its one-step approximation only for stochastic ordinary differential equations, we propose a convergence theorem on the mean-square orders of a class of semidiscrete schemes for the stochastic Schrödinger equation allowing sufficient spatial regularity. As a special case of stochastic symplectic Runge–Kutta methods applied to temporal discretization, the mean-square convergence order of a midpoint scheme is analyzed. Based on the convergence theorem, it is shown that the mean-square convergence order of the stochastic midpoint scheme is 1 under appropriate assumptions.

In section 2, we present some preliminaries on the stochastic Schrödinger equation. We derive the general formula of an infinite-dimensional stochastic Hamiltonian system via a variational principle with a stochastic forcing and show that the phase flow of the stochastic Schrödinger equation preserves symplecticity. In section 3, we propose a general class of stochastic symplectic Runge–Kutta methods in the temporal direction for the stochastic Schrödinger equation and show that they also preserve the charge conservation law. The midpoint scheme as an example of stochastic symplectic Runge–Kutta methods in the temporal direction is analyzed. In section 4 a convergence theorem on the mean-square orders of a class of semidiscrete schemes for the stochastic Schrödinger equation is proposed. Furthermore we obtain the mean-square convergence order of the midpoint scheme as an application of the theorem. At last, in section 5, numerical experiments are provided to validate theoretical results.

2. Stochastic Schrödinger equation. We are interested in the one-dimensional stochastic Schrödinger equation with multiplicative noise in the sense of Stratonovich

in the domain $[-L, L] \times [t_0, T]$,

$$(1) \qquad \mathbf{i}d_t\psi(x,t) + \Big(\frac{\partial^2 \psi}{\partial x^2} + \Psi'_{|\psi|^2}(|\psi|^2, x, t)\psi(x,t)\Big)dt = \varepsilon\psi(x,t) \circ dW(t),$$
$$\psi(-L, t) = \psi(L, t) = 0, \qquad \psi(x, t_0) = \varphi(x),$$

where $\mathbf{i}$ is the imaginary unit, the solution ψ is a $\mathbb{C}$-valued random field, $\frac{\partial^2 \psi}{\partial x^2}$ means the second derivative of function ψ with respect to variable x, $\Psi(|\psi|^2, x, t)$ is a real function of (ψ, x, t), $\Psi'_{|\psi|^2}(|\psi|^2, x, t)$ means the derivative with respect to $|\psi|^2$, $\varepsilon \in \mathbb{R}$ is a parameter describing the scale of noise, and $W(t)$ is an infinite-dimensional $\mathbf{Q}$-Wiener process which will be specified below. Moreover, we assume that the solution of (1) exists globally (see [8] for a study of existence for the solution of the stochastic Schrödinger equation). All the analysis in this paper could go through to the d-dimensional problem.

We rewrite (1) as a stochastic evolution equation in abstract form,

$$(2) \qquad \mathbf{i}d\psi(t) + \big(A\psi(t) + F(\psi(t))\big)dt = G(\psi(t)) \circ dW(t), \qquad \psi(t_0) = \varphi,$$

where $A = \frac{\partial^2}{\partial x^2}$ with Dirichlet boundary condition, $F(\psi(t))(x) = \Psi'_{|\psi|^2}(|\psi|^2, x, t)\psi(x, t)$, and $\big(G(\psi(t))(u)\big)(x) = \varepsilon\psi(x, t)u(x)$. From [15] we know that (2) satisfies the condition of commutative noise in infinite dimension, which is

$$\Big(G'(\psi(t))\big(G(\psi(t))u\big)\Big)(\tilde{u})(x) = \Big(G'(\psi(t))\big(G(\psi(t))\tilde{u}\big)\Big)(u)(x) = \varepsilon^2\psi(x, t)u(x)\tilde{u}(x).$$

$W(t)$ denotes a $\mathbf{Q}$-Wiener process on the Hilbert space $U = \mathbb{L}^2([-L, L], \mathbb{R})$ which is the subspace of $U_c = \mathbb{L}^2([-L, L], \mathbb{C})$ consisting of real-valued square integrable functions. Here $\mathbf{Q} \in \mathcal{L}(U)$ is nonnegative, symmetric, and with finite trace. Let $\{e_k\}_{k \in \mathbb{N}}$ be an orthonormal basis of U consisting of eigenvectors of $\mathbf{Q}$. Then there is a sequence of independent real-valued Brownian motions $\{\beta_k\}_{k \in \mathbb{N}}$ on the probability space $(\Omega, \mathcal{F}, P)$ such that $W(t) = \sum_{k \in \mathbb{N}} \beta_k(t)\mathbf{Q}^{\frac{1}{2}}e_k$, with covariance operator $\mathbf{Q} = \mathbf{Q}^{\frac{1}{2}} \circ \mathbf{Q}^{\frac{1}{2}}$; we refer to [20, 21] for more information about the infinite-dimensional Wiener process.

In addition, $\mathbf{Q}^{\frac{1}{2}}$ is a Hilbert–Schmidt operator from U to U (the space of all Hilbert–Schmidt operators from U to another Hilbert space H is denoted by $HS(U, H)$). In fact, $|\mathbf{Q}^{\frac{1}{2}}|_{HS(U,U)} = \sum_{k \in \mathbf{N}}\langle\mathbf{Q}^{\frac{1}{2}}e_k, \mathbf{Q}^{\frac{1}{2}}e_k\rangle_U = \sum_{k \in \mathbf{N}}\langle\mathbf{Q}e_k, e_k\rangle_U = tr(\mathbf{Q})$. More assumptions on $\mathbf{Q}$ will be specified later.

Here we mention an important property of the infinite-dimensional stochastic integral, which contributes a lot in the mean-square error estimation: for any predictable stochastic process G_s satisfying $\mathbf{E}\int_{t_0}^T |G_s\mathbf{Q}^{\frac{1}{2}}|^2_{HS(U,H)}ds < \infty$, we have

$$(3) \qquad \mathbf{E}\Big|\int_{t_0}^T G_s dW_s\Big|^2_H = \mathbf{E}\int_{t_0}^T |G_s\mathbf{Q}^{\frac{1}{2}}|^2_{HS(U,H)}ds,$$

where the stochastic integral on the left-hand side is of Itô sense and $\mathbf{E}(\cdot)$ means the expectation.

As pointed out in [8], there is an equivalent Itô form of (2),

$$(4) \qquad \mathbf{i}d\psi + (A\psi + \tilde{F}(\psi))dt = G(\psi)dW(t), \quad \psi(t_0) = \varphi,$$

where $\tilde{F}(\psi) = F(\psi) + \frac{\mathbf{i}}{2}\varepsilon^2\psi\aleph_{\mathbf{Q}}$ with $\aleph_{\mathbf{Q}}(x) = \sum_{k \in \mathbb{N}}(\mathbf{Q}^{\frac{1}{2}}e_k(x))^2$, for $x \in [-L, L]$. Here the series for the function $\aleph_{\mathbf{Q}}(x)$ converges.

The mild solution of (4) reads

$$(5) \qquad \psi(t) = S(t)\varphi + \mathbf{i} \int_{t_0}^{t} S(t-r)\tilde{F}(\psi(r))dr - \mathbf{i} \int_{t_0}^{t} S(t-r)G(\psi(r))dW(r),$$

where $S(t)_{t \in \mathbb{R}}$ denotes the group of solution operators of the deterministic linear differential equation $\mathbf{i}\frac{d\psi}{dt} = \Delta\psi$. We may use both (1) and (4) in the following analysis depending on which type of equation could provide much convenience; it is more convenient to use the equation in the Stratonovich sense in dealing with symplecticity, while the Itô equation provides much convenience when we make the error estimates.

If the size ε of the noise equals to 0, i.e., the noise term is eliminated, we get the deterministic Schrödinger equation. In this case, it possesses some global conservation laws, for example, charge conservation law $\mathcal{E}_1(\psi) = \int_{-L}^{L} |\psi|^2 dx = \mathcal{E}_1(\varphi)$, and energy conservation law if Ψ is independent of t, $\mathcal{E}_2(\psi) = \int_{-L}^{L}(|\psi_x|^2 - \Psi(|\psi|^2, x))dx = \mathcal{E}_2(\varphi)$. We refer to [12, 14] for further understanding. These quantities are important criteria of measuring whether a numerical simulation is good.

In the Stratonovich sense, the charge conservation law is still preserved by the solution of (1). In general, there is no energy conservation law for the stochastic Schrödinger equation. One only can obtain the relationship satisfied by the averaged energy [8, Proposition 4.5]. They are stated as follows.

PROPOSITION 1 (see [8]). *The stochastic nonlinear Schrödinger equation* (1) *possesses the charge conservation law P-a.s.,* $\mathcal{E}_1(\psi(t)) = \int_{-L}^{L} |\psi(t)|^2 dx = \mathcal{E}_1(\varphi) \; \forall t \in (t_0, T]$.

PROPOSITION 2 (see [8]). *The averaged energy* $\mathbf{E}(\mathcal{E}_2(\psi(t)))$ *satisfies*

$$\mathbf{E}(\mathcal{E}_2(\psi(t))) = \mathbf{E}(\mathcal{E}_2(\varphi)) + \frac{\varepsilon^2}{2} \int_{t_0}^{t} \int_{-L}^{L} |\psi(x,s)|^2 \sum_{k \in \mathbf{N}} \left| \frac{\partial}{\partial x} \mathbf{Q}^{\frac{1}{2}} e_k(x) \right|^2 dx ds$$

if Ψ *is independent of* t.

2.1. Preserving symplectic structure. One of the inherent canonical properties of the Hamiltonian system is the symplecticity of its flow (cf. [2]). The analysis of symplecticity of the stochastic Hamiltonian system is presented in [5, 19], for instance. As we all know, the deterministic Schrödinger equation can be rewritten as an infinite-dimensional Hamiltonian system with a classical Hamiltonian structure [9]. In this part, we will focus on the corresponding symplectic analysis of the stochastic Schrödinger equation. First, we present the general formula of infinite-dimensional stochastic Hamiltonian system via a variational principle with a stochastic forcing; see [26, Chapter 4.1] for the application of the generalized variational principle to the stochastic ordinary system.

Given two functionals $\mathcal{L}$ and $\mathcal{H}_2$, we define the generalized action integral by

$$\bar{S} = \int_{t_0}^{T} \Big(\mathcal{L}(P, Q, \dot{P}, \dot{Q}) - \mathcal{H}_2(P, Q) \circ \dot{\chi} \Big) dt,$$

where functions $P, Q : [-L, L] \times [t_0, T] \to \mathbb{R}$ and $\dot{P}, \dot{Q}$ mean the derivatives with respect to time. By denoting $\dot{\chi}$ the time-space noise, which is considered as the temporal derivative of the infinite-dimensional Wiener process $W(t)$, i.e., $\dot{\chi} = \frac{dW(t)}{dt}$, we may consider $\mathcal{H}_2 \circ \dot{\chi}$ formally as the work done by noise (nonconservative part).

The variation of the action integral follows as

$$
\begin{aligned}
\delta\bar{\mathcal{S}} &= \delta\int_{t_0}^{T}\Big(\mathcal{L}(P,Q,\dot{P},\dot{Q}) - \mathcal{H}_2(P,Q)\circ\dot{\chi}\Big)dt\\
&= \int_{-L}^{L}\int_{t_0}^{T}\Big(\frac{\delta\mathcal{L}}{\delta P}\delta P + \frac{\delta\mathcal{L}}{\delta Q}\delta Q + \frac{\delta\mathcal{L}}{\delta\dot{P}}\delta\dot{P} + \frac{\delta\mathcal{L}}{\delta\dot{Q}}\delta\dot{Q} - \frac{\delta\mathcal{H}_2}{\delta P}\delta P\circ\dot{\chi} - \frac{\delta\mathcal{H}_2}{\delta Q}\delta Q\circ\dot{\chi}\Big)dtdx\\
&= \int_{-L}^{L}\int_{t_0}^{T}\Big[\Big(\frac{\delta\mathcal{L}}{\delta P} - \frac{d}{dt}\Big(\frac{\delta\mathcal{L}}{\delta\dot{P}}\Big) - \frac{\delta\mathcal{H}_2}{\delta P}\circ\dot{\chi}\Big)\delta P + \Big(\frac{\delta\mathcal{L}}{\delta Q} - \frac{d}{dt}\Big(\frac{\delta\mathcal{L}}{\delta\dot{Q}}\Big) - \frac{\delta\mathcal{H}_2}{\delta Q}\circ\dot{\chi}\Big)\delta Q\Big]dtdx.
\end{aligned}
$$

Here an integration by parts is performed and the boundary conditions $\delta P(x,T) = \delta P(x,t_0) = 0$, $\delta Q(x,T) = \delta Q(x,t_0) = 0$ are used.

Thus the Hamilton's principle

$$
\delta\bar{\mathcal{S}} = \delta\int_{t_0}^{T}\Big(\mathcal{L}(P,Q,\dot{P},\dot{Q}) - \mathcal{H}_2(P,Q)\circ\dot{\chi}\Big)dt = 0
$$

leads to $\frac{d}{dt}\big(\frac{\delta\mathcal{L}}{\delta P}\big) = \frac{\delta\mathcal{L}}{\delta P} - \frac{\delta\mathcal{H}_2}{\delta P}\circ\dot{\chi}$, $\frac{d}{dt}\big(\frac{\delta\mathcal{L}}{\delta Q}\big) = \frac{\delta\mathcal{L}}{\delta Q} - \frac{\delta\mathcal{H}_2}{\delta Q}\circ\dot{\chi}$. Introduce the Hamiltonian $\mathcal{H}_1(P,Q)$ and the generator $\mathcal{G}$ which are defined according to the equation

$$
\mathcal{L}(P,Q,\dot{P},\dot{Q}) - \int_{-L}^{L}P\dot{Q}dx + \mathcal{H}_1(P,Q) - \frac{d\mathcal{G}}{dt} = 0.
$$

As in [26], using this equation, we find $\dot{P} = -\frac{\delta\mathcal{H}_1}{\delta Q} - \frac{\delta\mathcal{H}_2}{\delta Q}\circ\dot{\chi}$, $\dot{Q} = \frac{\delta\mathcal{H}_1}{\delta P} + \frac{\delta\mathcal{H}_2}{\delta P}\circ\dot{\chi}$, which can be rewritten in the form

$$
\text{(6)}\qquad dP = -\frac{\delta\mathcal{H}_1}{\delta Q}dt - \frac{\delta\mathcal{H}_2}{\delta Q}\circ dW(t),\qquad dQ = \frac{\delta\mathcal{H}_1}{\delta P}dt + \frac{\delta\mathcal{H}_2}{\delta P}\circ dW(t).
$$

Remark 3. If $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$, then $\mathcal{H}$ is an invariant of system (6). In fact,

$$
\begin{aligned}
d\mathcal{H} &= \int_{-L}^{L}\Big(\frac{\delta\mathcal{H}}{\delta P}dP + \frac{\delta\mathcal{H}}{\delta Q}dQ\Big)dx\\
&= \int_{-L}^{L}\Big(-\frac{\delta\mathcal{H}}{\delta P}\frac{\delta\mathcal{H}}{\delta Q}dt - \frac{\delta\mathcal{H}}{\delta P}\frac{\delta\mathcal{H}}{\delta Q}\circ dW + \frac{\delta\mathcal{H}}{\delta Q}\frac{\delta\mathcal{H}}{\delta P}dt + \frac{\delta\mathcal{H}}{\delta Q}\frac{\delta\mathcal{H}}{\delta P}\circ dW\Big)dx = 0.
\end{aligned}
$$

The stochastic Schrödinger equation (1) can be written as a canonical infinite-dimensional Hamiltonian system. Denote the real and imaginary parts of the solution of stochastic Schrödinger equation (1) by $\Re(\psi(x,t)) = P(x,t)$ and $\Im(\psi(x,t)) = Q(x,t)$, respectively. Then we have

(7)

$$
dP = -\frac{\delta\mathcal{H}_1}{\delta Q}dt - \frac{\delta\mathcal{H}_2}{\delta Q}\circ dW(t) = -(AQ + \Psi'_{|\psi|^2}(P^2 + Q^2,x,t)Q)dt + \varepsilon Q\circ dW(t),
$$

$$
dQ = \frac{\delta\mathcal{H}_1}{\delta P}dt + \frac{\delta\mathcal{H}_2}{\delta P}\circ dW(t) = (AP + \Psi'_{|\psi|^2}(P^2 + Q^2,x,t)P)dt - \varepsilon P\circ dW(t)
$$

with initial conditions $P(0) = p = \Re(\varphi)$, $Q(0) = q = \Im(\varphi)$, and Hamiltonians $\mathcal{H}_1(P,Q) = \frac{1}{2}\int_{-L}^{L}((P_x^2 + Q_x^2) - \Psi(P^2 + Q^2,x,t))dx$ and $\mathcal{H}_2(P,Q) = -\frac{\varepsilon}{2}\int_{-L}^{L}(P^2 + Q^2)dx$.

The symplectic form for system (7) is given by

$$
\text{(8)}\qquad \bar{\omega}(t) = \int_{-L}^{L}dP\wedge dQ\,dx,
$$

where the overbar on ω is a reminder that the two-form $dP \wedge dQ$ is integrated over the space. Preservation of the symplectic form (8) means that the spatial integral of the oriented areas of projections onto the coordinate planes (p, q) is an integral invariant [2]. Note that the differentials in (7) and (8) have different meanings. In (7), P, Q are treated as functions of time, and p, q are fixed functions, while in (8) the differential is made with respect to initial data p, q.

Using the formula of change of variables in differential forms, we obtain

$$\bar{\omega}(t) = \int_{-L}^{L} dP \wedge dQ \, dx = \int_{-L}^{L} \Big(\frac{\partial P}{\partial p} \frac{\partial Q}{\partial q} - \frac{\partial P}{\partial q} \frac{\partial Q}{\partial p} \Big) dp \wedge dq \, dx.$$

Hence, the phase flow of (7) preserves symplectic structure if and only if

$$\frac{d\bar{\omega}(t)}{dt} = \int_{-L}^{L} \frac{d}{dt} \Big(\frac{\partial P}{\partial p} \frac{\partial Q}{\partial q} - \frac{\partial P}{\partial q} \frac{\partial Q}{\partial p} \Big) dp \wedge dq \, dx = 0.$$

We introduce the notation $P_p = \frac{\partial P}{\partial p}$, $P_q = \frac{\partial P}{\partial q}$, $Q_p = \frac{\partial Q}{\partial p}$, $Q_q = \frac{\partial Q}{\partial q}$ and let $H_1(P, Q) = \frac{1}{2}\Psi(P^2 + Q^2, x, t)$. We know that from the differentiability with respect to initial data of stochastic infinite-dimensional equations, P_p, Q_p, P_q, and Q_q obey the following system (see [13, Chapter 4] and [20, Chapter 9]):

$$(9) \quad dP_p = -\Big(AQ_p + \frac{\partial^2 H_1}{\partial P \partial Q} P_p + \frac{\partial^2 H_1}{\partial Q^2} Q_p \Big) dt + \varepsilon Q_p \circ dW(t), \quad P_p(t_0) = I,$$

$$dQ_p = \Big(AP_p + \frac{\partial^2 H_1}{\partial P^2} P_p + \frac{\partial^2 H_1}{\partial P \partial Q} Q_p \Big) dt - \varepsilon P_p \circ dW(t), \quad Q_p(t_0) = 0,$$

$$dP_q = -\Big(AQ_q + \frac{\partial^2 H_1}{\partial P \partial Q} P_q + \frac{\partial^2 H_1}{\partial Q^2} Q_q \Big) dt + \varepsilon Q_q \circ dW(t), \quad P_q(t_0) = 0,$$

$$dQ_q = \Big(AP_q + \frac{\partial^2 H_1}{\partial P^2} P_q + \frac{\partial^2 H_1}{\partial P \partial Q} Q_q \Big) dt - \varepsilon P_q \circ dW(t), \quad Q_q(t_0) = I.$$

Due to (9), we get

$$d\Big(\frac{\partial P}{\partial p} \frac{\partial Q}{\partial q} - \frac{\partial P}{\partial q} \frac{\partial Q}{\partial p} \Big) = d\Big(P_p Q_q - P_q Q_p \Big) = d(P_p)Q_q + d(Q_q)P_p - d(P_q)Q_p - d(Q_p)P_q$$

$$= \Big[-\Big(AQ_p + \frac{\partial^2 H_1}{\partial P \partial Q} P_p + \frac{\partial^2 H_1}{\partial Q^2} Q_p \Big) Q_q + \Big(AQ_q + \frac{\partial^2 H_1}{\partial P \partial Q} P_q + \frac{\partial^2 H_1}{\partial Q^2} Q_q \Big) Q_p$$

$$+ \Big(AP_q + \frac{\partial^2 H_1}{\partial P^2} P_q + \frac{\partial^2 H_1}{\partial P \partial Q} Q_q \Big) P_p - \Big(AP_p + \frac{\partial^2 H_1}{\partial P^2} P_p + \frac{\partial^2 H_1}{\partial P \partial Q} Q_p \Big) P_q \Big] dt$$

$$+ \varepsilon \Big(Q_p Q_q - Q_q Q_p - P_p P_q + P_q P_p \Big) \circ dW(t)$$

$$= \Big(-(AQ_p)Q_q + (AQ_q)Q_p + (AP_q)P_p - (AP_p)P_q \Big) dt.$$

Note that A is Laplacian; then we have

$$\frac{d}{dt}\bar{\omega} = \int_{-L}^{L} \frac{d}{dt} \Big(\frac{\partial P}{\partial p} \frac{\partial Q}{\partial q} - \frac{\partial P}{\partial q} \frac{\partial Q}{\partial p} \Big) dp \wedge dq \, dx$$

$$= \int_{-L}^{L} \Big(-AQ_p Q_q + AQ_q Q_p + AP_q P_p - AP_p P_q \Big) dp \wedge dq \, dx$$

$$= -\int_{-L}^{L} [d(AQ) \wedge dQ + d(AP) \wedge dP] \, dx = -\int_{-L}^{L} \frac{\partial}{\partial x} [d(Q_x) \wedge dQ + d(P_x) \wedge dP] dx.$$

From the zero boundary condition, we obtain that $\frac{d}{dt}\bar{\omega}(t) = 0$. Thus we have the following theorem.

THEOREM 4. *The phase flow of the stochastic Schrödinger equation*

$$\mathbf{i}d\psi + (A\psi + F(\psi))dt = G(\psi) \circ dW,$$

with $F(\psi) = \Psi'_{|\psi|^2}(|\psi|^2, x, t)\psi$ and $G(\psi) = \varepsilon\psi$, preserves the symplectic structure

$$\bar{\omega}(t) = \int_{-L}^{L} dP \wedge dQ\ dx$$

with P (resp., Q) being the real (resp., imaginary) part of the solution ψ.

3. Symplectic Runge–Kutta semidiscretization in the temporal direction. For the stochastic ordinary differential equation, the stochastic Runge–Kutta method is an important class of numerical methods, which has been widely investigated; see [10, 18, 19, 23, 25] and references therein, for instance.

Here we introduce the uniform partition $0 = t_0 < t_1 < \cdots < t_N = T$ for simplicity, let $\Delta t = t_{n+1} - t_n$, $n = 0, 1, \ldots, N - 1$, and $\Delta W_n = W(t_{n+1}) - W(t_n)$. Applying s-stage stochastic Runge–Kutta methods, which only depend on the increments of the Wiener process, to (7) in the temporal direction, where P and Q mean the real and imaginary parts of solution for stochastic Schrödinger equation, respectively, we get that

$$(10a) \qquad P_i = P^n - \Delta t \sum_{j=1}^{s} a_{ij}^{(0)}(AQ_j + \Psi'_j Q_j) + \Delta W_n \sum_{j=1}^{s} a_{ij}^{(1)}\varepsilon Q_j,$$

$$(10b) \qquad Q_i = Q^n + \Delta t \sum_{j=1}^{s} a_{ij}^{(0)}(AP_j + \Psi'_j P_j) - \Delta W_n \sum_{j=1}^{s} a_{ij}^{(1)}\varepsilon P_j,$$

$$(10c) \qquad P^{n+1} = P^n - \Delta t \sum_{i=1}^{s} b_i^{(0)}(AQ_i + \Psi'_i Q_i) + \Delta W_n \sum_{i=1}^{s} b_i^{(1)}\varepsilon Q_i,$$

$$(10d) \qquad Q^{n+1} = Q^n + \Delta t \sum_{i=1}^{s} b_i^{(0)}(AP_i + \Psi'_i P_i) - \Delta W_n \sum_{i=1}^{s} b_i^{(1)}\varepsilon P_i,$$

where $\Psi'_i = \Psi'_{|\psi|^2}(P_i^2 + Q_i^2, x, t_n + \sum_{j=1}^{s} a_{ij}^{(0)}\Delta t)$ for $i = 1, \ldots, s$ and $n = 0, \ldots, N-1$, $\left(a_{ij}^{(0)}\right)_{s \times s}$ and $\left(a_{ij}^{(1)}\right)_{s \times s}$ are $s \times s$ matrices of real elements while $\left(b_1^{(0)}, \ldots, b_s^{(0)}\right)$ and $\left(b_1^{(1)}, \ldots, b_s^{(1)}\right)$ are real vectors.

Under the assumption of existence of solution of (10), we could obtain the following condition for (10) to preserve the discrete symplectic structure. We postpone the existence of solution to Proposition 6.

THEOREM 5. *Methods (10) preserve the discrete symplectic structure, i.e.,*

$$\bar{\omega}^{n+1} = \int_{-L}^{L} dP^{n+1} \wedge dQ^{n+1}\ dx = \int_{-L}^{L} dP^n \wedge dQ^n\ dx = \bar{\omega}^n,$$

if coefficients satisfy the following conditions: $\forall\ i, j = 1, \ldots, s$,

$$(11)$$
$$b_i^{(0)}b_j^{(0)} = b_i^{(0)}a_{ij}^{(0)} + b_j^{(0)}a_{ji}^{(0)}, \quad b_i^{(0)}b_j^{(1)} = b_i^{(0)}a_{ij}^{(1)} + b_j^{(1)}a_{ji}^{(0)}, b_i^{(1)}b_j^{(1)} = b_i^{(1)}a_{ij}^{(1)} + b_j^{(1)}a_{ji}^{(1)}.$$

Proof. From (10c) and (10d), we have

$$(12) \quad dP^{n+1} \wedge dQ^{n+1} - dP^n \wedge dQ^n$$

$$= -\Delta W_n^2 \sum_{i,j=1}^s b_i^{(1)} b_j^{(1)} d\tilde{f}_i \wedge \tilde{g}_j - \Delta W_n \sum_{i=1}^s b_i^{(1)} \Big(dP^n \wedge d\tilde{g}_i - d\tilde{f}_i \wedge dQ^n \Big)$$

$$+ \Delta t \sum_{i=1}^s b_i^{(0)} \Big(dP^n \wedge d(AP_i + g_i) - d(AQ_i + f_i) \wedge dQ^n \Big)$$

$$+ \Delta t \Delta W_n \sum_{i,j=1}^s b_i^{(0)} b_j^{(1)} \Big(d(AQ_i + f_i) \wedge d\tilde{g}_j + d\tilde{f}_j \wedge d(AP_i + g_i) \Big)$$

$$- \Delta t^2 \sum_{i,j=1}^s b_i^{(0)} b_j^{(0)} \Big(d(AQ_i) \wedge d(AP_j + g_j) + df_i \wedge d(AP_j + g_j) \Big),$$

where $f_j = \Psi_j' Q_j$, $\tilde{f}_j = \varepsilon Q_j$, $g_j = \Psi_j' P_j$, $\tilde{g}_j = \varepsilon P_j$. And from (10a) and (10b),

$$(13) \quad dP^n = dP_i + \Delta t \sum_{j=1}^s a_{ij}^{(0)} d(AQ_j) + \Delta t \sum_{j=1}^s a_{ij}^{(0)} df_j - \Delta W_n \sum_{j=1}^s a_{ij}^{(1)} d\tilde{f}_j,$$

$$dQ^n = dQ_i - \Delta t \sum_{j=1}^s a_{ij}^{(0)} d(AP_j) - \Delta t \sum_{j=1}^s a_{ij}^{(0)} dg_j + \Delta W_n \sum_{j=1}^s a_{ij}^{(1)} d\tilde{g}_j.$$

Substituting (13) into (12), we obtain

$$dP^{n+1} \wedge dQ^{n+1} - dP^n \wedge dQ^n = \Delta W_n \sum_{i=1}^s b_i^{(1)} \Big(d\tilde{f}_i \wedge dQ_i - dP_i \wedge d\tilde{g}_i \Big)$$

$$+ \Delta t \sum_{i=1}^s b_i^{(0)} \Big(dP_i \wedge d(AP_i) + dP_i \wedge dg_i - d(AQ_i) \wedge dQ_i - df_i \wedge dQ_i \Big)$$

$$+ \Delta W_n^2 \sum_{i,j=1}^s (b_i^{(1)} a_{ij}^{(1)} + b_j^{(1)} a_{ji}^{(1)} - b_i^{(1)} b_j^{(1)}) d\tilde{f}_i \wedge \tilde{g}_j$$

$$+ \Delta t \Delta W_n \sum_{i,j=1}^s (b_i^{(0)} b_j^{(1)} - b_i^{(0)} a_{ij}^{(1)} - b_j^{(1)} a_{ji}^{(0)})$$

$$\Big(d(AQ_i) \wedge d\tilde{g}_j + d\tilde{f}_i \wedge d\tilde{g}_j + d\tilde{f}_j \wedge d(AP_i) + d\tilde{f}_j \wedge dg_i \Big)$$

$$+ \Delta t^2 \sum_{i,j=1}^s (b_i^{(0)} a_{ij}^{(0)} + b_j^{(0)} a_{ji}^{(0)} - b_i^{(0)} b_j^{(0)})$$

$$\Big(d(AQ_i \wedge d(AP_j) + d(AQ_i) \wedge dg_j + df_i \wedge d(AP_j) + df_i \wedge dg_j) \Big).$$

Recalling the conditions (11) and the expressions of functions f_i, $\tilde{f}_i$, g_i, $\tilde{g}_i$, we have

$$\int_{-L}^{L} dP^{n+1} \wedge dQ^{n+1} \, dx = \int_{-L}^{L} dP^n \wedge dQ^n \, dx$$

$$+ \Delta t \sum_{i=1}^s b_i^{(0)} \int_{-L}^{L} \frac{\partial}{\partial x} (dP_i \wedge d(P_i)_x - d(Q_i)_x \wedge dQ_i) \, dx = \int_{-L}^{L} dP^n \wedge dQ^n \, dx.$$

Thus the proof of the theorem is completed. □

The solution of stochastic symplectic Runge–Kutta methods (10) exists in space $U_c = \mathbb{L}^2([-L, L], \mathbb{C})$, and it preserves the discrete charge conservation law.

PROPOSITION 6. *There exists a U_c-valued solution of the symplectic methods* (10)– (11). *Moreover, it possesses the discrete charge conservation law, i.e.,*

$$\int_{-L}^{L} |\phi^{n+1}|^2 dx = \int_{-L}^{L} |\phi^n|^2 dx \quad \forall n = 0, 1, \ldots, N.$$

Proof. First, we show the preservation of charge conservation law. Recall that $\phi^n = P^n + \mathbf{i}Q^n$ and let $\phi_i = P_i + \mathbf{i}Q_i$; then methods (10) can be rewritten as

$$(14a) \qquad \phi_i = \phi^n + \mathbf{i}\Delta t \sum_{j=1}^{s} a_{ij}^{(0)} h_j - \mathbf{i}\Delta W_n \sum_{j=1}^{s} a_{ij}^{(1)} \tilde{h}_j,$$

$$(14b) \qquad \phi^{n+1} = \phi^n + \mathbf{i}\Delta t \sum_{i=1}^{s} b_i^{(0)} h_i - \mathbf{i}\Delta W_n \sum_{i=1}^{s} b_i^{(1)} \tilde{h}_i$$

with $h_i = A\phi_i + \Psi_i'\phi_i$ and $\tilde{h}_i = \varepsilon\phi_i$.

Multiplying (14b) with $\bar{\phi}^{n+1} + \bar{\phi}^n$ and taking real part, we get

(15)

$$|\phi^{n+1}|^2 = |\phi^n|^2 + \Re\Big\{\mathbf{i}\Delta t \sum_{i=1}^{s} b_i^{(0)} h_i(\bar{\phi}^{n+1} + \bar{\phi}^n) - \mathbf{i}\Delta W_n \sum_{i=1}^{s} b_i^{(1)} \tilde{h}_i(\bar{\phi}^{n+1} + \bar{\phi}^n)\Big\}.$$

From (14a), we have the expression of ϕ^n,

$$\phi^n = \phi_i - \mathbf{i}\Delta t \sum_{j=1}^{s} a_{ij}^{(0)} h_j + \mathbf{i}\Delta W_n \sum_{j=1}^{s} a_{ij}^{(1)} \tilde{h}_j.$$

Combining this together with (14b) leads to

$$\bar{\phi}^{n+1} + \bar{\phi}^n = 2\bar{\phi}_i + \mathbf{i}2\Delta t \sum_{j=1}^{s} a_{ij}^{(0)} \bar{h}_j - \mathbf{i}2\Delta W_n \sum_{j=1}^{s} a_{ij}^{(1)} \bar{\tilde{h}}_j$$

$$- \mathbf{i}\Delta t \sum_{j=1}^{s} b_j^{(0)} \bar{h}_j + \mathbf{i}\Delta W_n \sum_{j=1}^{s} b_j^{(1)} \bar{\tilde{h}}_j.$$

Therefore the second term of the right-hand side of (15) becomes

$$\Re\Big\{\mathbf{i}\Delta t \sum_{i=1}^{s} b_i^{(0)} h_i(\bar{\phi}^{n+1} + \bar{\phi}^n) - \mathbf{i}\Delta W_n \sum_{i=1}^{s} b_i^{(1)} \tilde{h}_i(\bar{\phi}^{n+1} + \bar{\phi}^n)\Big\}$$

$$= \Im\Big\{2\Delta t \sum_{i=1}^{s} b_i^{(0)} A\phi_i \bar{\phi}_i\Big\} + \Delta t^2 \sum_{ij=1}^{s} \Big(b_i^{(0)} b_j^{(0)} - b_i^{(0)} a_{ij}^{(0)} - b_j^{(0)} a_{ji}^{(0)}\Big)\Re(h_i\bar{h}_j)$$

$$- 2\Delta t\Delta W_n \sum_{ij=1}^{s} \Big(b_i^{(0)} b_j^{(1)} - b_i^{(0)} a_{ij}^{(1)} - b_j^{(1)} a_{ji}^{(0)}\Big)\Re(h_i\bar{\tilde{h}}_j)$$

$$+ \Delta W_n^2 \sum_{ij=1}^{s} \Big(b_i^{(1)} b_j^{(1)} - b_i^{(1)} a_{ij}^{(1)} - b_j^{(1)} a_{ji}^{(1)}\Big)\Re(\tilde{h}_i\bar{\tilde{h}}_j),$$

since $\Re\phi = \Re\bar{\phi}$. Recalling the symplectic condition (11), we have

$$|\phi^{n+1}|^2 = |\phi^n|^2 + \Im\Big\{2\Delta t \sum_{i=1}^s b_i^{(0)} A\phi_i \bar{\phi}_i\Big\}.$$

Integrating the above equation from $-L$ to L with respect to x leads to the preservation of charge conservation law.

Second, the existence of solution follows from a standard Galerkin method and Brouwer's theorem; see [1, Lemma 3.1] and [6, Lemma 3.1]. From (14b), we have that

$$\phi^{n+\frac{1}{2}} = \phi^n + \frac{\mathbf{i}}{2}\Delta t \sum_{i=1}^s b_i^{(0)} h_i - \frac{\mathbf{i}}{2}\Delta W_n \sum_{i=1}^s b_i^{(1)} \tilde{h}_i.$$

Thus define the mapping $\Pi : U_c \to U_c$ by

$$\Pi(\phi^{n+\frac{1}{2}}) = \phi^{n+\frac{1}{2}} - \phi^n - \frac{\mathbf{i}}{2}\Delta t \sum_{i=1}^s b_i^{(0)} h_i + \frac{\mathbf{i}}{2}\Delta W_n \sum_{i=1}^s b_i^{(1)} \tilde{h}_i.$$

A similar proof leads to

$$\Re\langle\Pi(\phi^{n+\frac{1}{2}}), \phi^{n+\frac{1}{2}}\rangle_{U_c} = \|\phi^{n+\frac{1}{2}}\|_{U_c}^2 - \Re\langle\phi^n, \phi^{n+\frac{1}{2}}\rangle_{U_c} \geq \|\phi^{n+\frac{1}{2}}\|_{U_c}\big(\|\phi^{n+\frac{1}{2}}\|_{U_c} - \|\phi^n\|_{U_c}\big).$$

Hence for every $\phi^{n+\frac{1}{2}} \in U_c$ such that $\|\phi^{n+\frac{1}{2}}\|_{U_c} = \|\phi^n\|_{U_c} + 1$, there holds $\Re\langle\Pi(\phi^{n+\frac{1}{2}}), \phi^{n+\frac{1}{2}}\rangle_{U_c} > 0$. The existence of solution follows from a standard Galerkin method (make the above argument rigorous) and the Brouwer's theorem. $\square$

3.1. Midpoint scheme. In what follows, we consider a special case of symplectic Runge–Kutta methods. Let $s = 1$ and take coefficients as $a_{11}^{(0)} = a_{11}^{(1)} = \frac{1}{2}$ and $b_1^{(0)} = b_1^{(1)} = 1$. Obviously, coefficients satisfy symplectic conditions (11), and we obtain the midpoint scheme:

$$(16) \qquad P^{n+1} = P^n - \Delta t A Q^{n+\frac{1}{2}} - \Delta t (\Psi')^{n+\frac{1}{2}} Q^{n+\frac{1}{2}} + \varepsilon Q^{n+\frac{1}{2}} \Delta W_n,$$
$$Q^{n+1} = Q^n + \Delta t A P^{n+\frac{1}{2}} + \Delta t (\Psi')^{n+\frac{1}{2}} P^{n+\frac{1}{2}} - \varepsilon P^{n+\frac{1}{2}} \Delta W_n.$$

Here $P^{n+\frac{1}{2}} = \frac{1}{2}(P^n + P^{n+1})$ and $Q^{n+\frac{1}{2}} = \frac{1}{2}(Q^n + Q^{n+1})$. Denote the approximation of $\psi(t_n)$ by ϕ^n, and recall that $\phi^n = P^n + \mathbf{i}Q^n$ and $\phi^{n+\frac{1}{2}} = \frac{1}{2}(\phi^n + \phi^{n+1})$; we have

$$(17) \qquad \phi^{n+1} = \phi^n + \mathbf{i}\Delta t A \phi^{n+\frac{1}{2}} + \mathbf{i}\Delta t F(t_{n+\frac{1}{2}}, \phi^{n+\frac{1}{2}}) - \mathbf{i}\varepsilon \phi^{n+\frac{1}{2}} \Delta W_n.$$

Of course, (17) is formal and has to be understood in the following sense:

$$(18) \qquad \phi^{n+1} = \hat{S}_{\Delta t} \phi^n + \mathbf{i}\Delta t T_{\Delta t} F(t_{n+\frac{1}{2}}, \phi^{n+\frac{1}{2}}) - \mathbf{i}\varepsilon T_{\Delta t} \phi^{n+\frac{1}{2}} \Delta W_n,$$

where $F(t_{n+\frac{1}{2}}, \phi^{n+\frac{1}{2}}) := \Psi'_{|\psi|^2}(|\phi^{n+\frac{1}{2}}|^2, x, t_{n+\frac{1}{2}})\phi^{n+\frac{1}{2}}$, and the operators are

$$(19) \qquad \hat{S}_{\Delta t} = \left(Id - \frac{\mathbf{i}\Delta t}{2}A\right)^{-1}\left(Id + \frac{\mathbf{i}\Delta t}{2}A\right) \text{ and } T_{\Delta t} = \left(Id - \frac{\mathbf{i}\Delta t}{2}A\right)^{-1}.$$

Since the midpoint scheme is a special case of symplectic Runge–Kutta methods, it also preserves the discrete charge conservation law.

COROLLARY 7. *For midpoint scheme* (16), *the discrete charge conservation law is preserved, i.e., P-a.s.,*

$$\int_{-L}^{L} |\phi^{n+1}|^2 dx = \int_{-L}^{L} |\phi^n|^2 dx \quad \forall n = 0, 1, \ldots, N.$$

The following propositions are discrete versions of the evolutionary relationship of average energy for midpoint scheme. For the general Runge–Kutta methods, it is an open problem.

PROPOSITION 8. *If* $\Psi'_{|\psi|^2}(|\psi|^2, x, t) = V(x)$, *then*

$$\mathbf{E}(\mathcal{E}_2(\phi^{n+1})) + \frac{\varepsilon}{\Delta t} \int_{-L}^{L} \mathbf{E}(|\phi^{n+1}|^2 \Delta W_n) dx = \mathbf{E}(\mathcal{E}_2(\phi^n)),$$

where $\mathcal{E}_2(\phi^n) = \int_{-L}^{L} |\nabla\phi^n|^2 dx - \int_{-L}^{L} V(x)|\phi^n|^2 dx.$

Proof. Since scheme (18) can be rewritten as

$$\frac{\mathbf{i}}{\Delta t}(\phi^{n+1} - \phi^n) + A\phi^{n+\frac{1}{2}} + \Psi'_{|\psi|^2}(|\phi^{n+\frac{1}{2}}|^2, x, t^{n+\frac{1}{2}})\phi^{n+\frac{1}{2}} = \frac{\varepsilon}{\Delta t}\phi^{n+\frac{1}{2}}\Delta W_n,$$

multiplying it with $\bar{\phi}^{n+1} - \bar{\phi}^n$, taking the real part, and integrating with respect to x in the space $[-L, L]$, the first term in the left-hand side of the above equation becomes

$$\Im \int_{-L}^{L} \frac{1}{\Delta t}(\phi^{n+1} - \phi^n)(\bar{\phi}^{n+1} - \bar{\phi}^n)dx = -\frac{1}{\Delta t}\Im \int_{-L}^{L}(\phi^{n+1}\bar{\phi}^n + \phi^n\bar{\phi}^{n+1})dx = 0,$$

the second term has the form

$$\Re \int_{-L}^{L} A\phi^{n+\frac{1}{2}}(\bar{\phi}^{n+1} - \bar{\phi}^n)dx = \frac{1}{2}\Re \int_{-L}^{L} \Big(-|\nabla\phi^{n+1}|^2 + |\nabla\phi^n|^2$$
$$+ \nabla\phi^{n+1}\nabla\bar{\phi}^n - \nabla\phi^n\nabla\bar{\phi}^{n+1}\Big)dx$$
$$= -\frac{1}{2}\int_{-L}^{L} |\nabla\phi^{n+1}|^2 dx + \frac{1}{2}\int_{-L}^{L} |\nabla\phi^n|^2 dx,$$

the third term acquires the form

$$\frac{1}{2}\Re \int_{-L}^{L} V(x)(\phi^{n+1} + \phi^n)(\bar{\phi}^{n+1} - \bar{\phi}^n)dx = \frac{1}{2}\int_{-L}^{L} V(x)|\phi^{n+1}|^2 dx - \frac{1}{2}\int_{-L}^{L} V(x)|\phi^n|^2 dx,$$

at last, the term turns into

$$\Re \int_{-L}^{L} \Big(\frac{\varepsilon}{\Delta t}\Delta W_n \phi^{n+\frac{1}{2}}(\bar{\phi}^{n+1} - \bar{\phi}^n)\Big)dx$$
$$= \frac{\varepsilon}{2\Delta t}\int_{-L}^{L} |\phi^{n+1}|^2 \Delta W_n dx - \frac{\varepsilon}{2\Delta t}\int_{-L}^{L} |\phi^n|^2 \Delta W_n dx.$$

Combining the above equations, the result of the proposition is obtained. □

PROPOSITION 9. *For the general* $\Psi'_{|\psi|^2}(|\psi|^2, x, t)$, *we have the implicit relationship*

$$\int_{-L}^{L} |\nabla\phi^{n+1}|dx - \int_{-L}^{L} |\nabla\phi^n|dx - \int_{-L}^{L} (\Psi')^{n+\frac{1}{2}}(|\phi^{n+1}|^2 - |\phi^n|^2)dx$$
$$= -\varepsilon \int_{-L}^{L} (|\phi^{n+1}|^2 - |\phi^n|^2)\Delta W_n dx.$$

Proof. The proof of this proposition is the same as that of Proposition 8.   □

4. Convergence theorem. To investigate the mean-square convergence order of semidiscrete approximations to the stochastic Schrödinger equation, we establish a convergence theorem on the relationship between local accuracy order and global mean-square order of a semidiscrete method.

In this section, we consider stochastic Schrödinger equation in the Itô sense,

$$\mathbf{i}d\psi + (A\psi + F(\psi))dt = G(\psi)dW, \tag{20}$$

whose mild solution is

$$\psi(t_{n+1}) = S(\Delta t)\psi(t_n) + \mathbf{i}\int_{t_n}^{t_{n+1}} S(t_{n+1} - r)F(\psi(r))dr$$

$$- \mathbf{i}\int_{t_n}^{t_{n+1}} S(t_{n+1} - r)G(\psi(r))dW(r)$$

$$:= S(\Delta t)\psi(t_n) + \Upsilon(\psi(r)).$$

Denote the approximation of the solution $(\psi(t_n), \mathcal{F}_{t_n})$ for (20) by $(\phi^n, \mathcal{F}_{t_n})$, which means that the numerical approximation ϕ^n is also $\mathcal{F}_{t_n}$-measurable. Here $\mathcal{F}_t$ is the filtration generated by the Wiener process and initial value. Define adapted numerical solution ϕ^n recurrently by

$$\phi^{n+1} = \hat{S}_{\Delta t}\phi^n + \Gamma(\phi^n, \phi^{n+1}, \Delta t, \Delta W_n, T_{\Delta t}), \tag{21}$$

where $\hat{S}_{\Delta t}$ and $T_{\Delta t}$ are two operators depending on operator A, such that $|\hat{S}_{\Delta t}|_{\mathcal{L}(U_c, U_c)} \leq 1$, which don't have to be the same as (19) as long as they satisfy the mentioned hypotheses.

Denote the usual Sobolev space $\mathbb{H}^s := \mathbb{H}^s([-L, L]; \mathbb{C})$ for $s \in \mathbb{R}$. We write $|\cdot|_s = |\cdot|_{\mathbb{H}^s}$ and make the following hypotheses for (20).

HYPOTHESIS 10. *For $\alpha \geq 0$, assume that $F : \mathbb{H}^\alpha \to \mathbb{H}^\alpha$ and $G(\cdot)\mathbf{Q}^{\frac{1}{2}} : \mathbb{H}^\alpha \to HS(U; \mathbb{H}^\alpha)$ are globally Lipschitz functions.*

Since the group $S(t)$ has no smooth effects on the solution, we assume that the solution has extra regularity. Let $\beta > \alpha$.

HYPOTHESIS 11. *The solution ψ is in $L^2(\Omega; L^2(0, T; \mathbb{H}^\beta))$, $F(\psi)$ is in $L^2(\Omega; L^2(0, T; \mathbb{H}^\beta))$, $G(\psi)\mathbf{Q}^{\frac{1}{2}}$ is in $L^2(\Omega; L^2(0, T; HS(U; \mathbb{H}^\beta)))$, and initial data $\varphi \in L^2(\Omega; \mathbb{H}^\beta)$.*

The difference of two operators $S(\Delta t)$ and $\hat{S}_{\Delta t}$ is essential in the mean-square convergence estimation. Furthermore, we assume that the order of approximation to the group $S(t)$ is of order q, which is stated below. Throughout this paper, all constants K depend on the coefficients of (20), the operator $\mathbf{Q}^{\frac{1}{2}}$, and initial value φ and do not depend on n and Δt, which may be different from line to line.

HYPOTHESIS 12. *There exist constants $K > 0$ and $q > 0$ such that*

$$|S(t_k) - \hat{S}_{\Delta t}^k|_{\mathcal{L}(\mathbb{H}^\beta, \mathbb{H}^\alpha)} \leq K\Delta t^q.$$

Milstein and Tretyakov [19] establish the mean-square order of convergence of a method resting on properties of its one-step approximation only for stochastic ordinary

differential equations. The most successful point in the proof of the fundamental convergence theorem is the usage of conditional expectation. Properties of stochastic partial differential equations are different from those of stochastic ordinary differential equations; for example, in the mild solution forms of the continuous and discrete solutions, there are two different deterministic operators $S(\Delta t)$ and $\hat{S}_{\Delta t}$. So we have to separate the operator term and the integral term. Here we solve it by introducing a temporary process $\tilde{\psi}(r)$, $t_n \leq r \leq t_{n+1}$, which is a continuous approximation of $\psi(r)$:

$$(22) \quad \tilde{\psi}(r) = \hat{S}_{r-t_n}\tilde{\psi}(t_n) + \mathbf{i}\int_{t_n}^{r} S(r-\rho)F(\tilde{\psi}(\rho))d\rho - \mathbf{i}\int_{t_n}^{r} S(r-\rho)G(\tilde{\psi}(\rho))dW(\rho),$$

$$\tilde{\psi}(x,0) = \varphi(x).$$

If $r = t_{n+1}$, $\tilde{\psi}(t_{n+1}) = \hat{S}_{\Delta t}\tilde{\psi}(t_n) + \Upsilon(\tilde{\psi}(r))$.

Let ϕ be an $\mathcal{F}_{t_n}$-measurable random variable. $\tilde{\psi}_{t_n,\phi}(t)$ denotes the solution of the system (22) for $t_n \leq t \leq T$ satisfying the initial condition at $t = t_n$: $\tilde{\psi}(t_n) = \phi$.

PROPOSITION 13. *There exists a function Z such that we have the representation*

$$\tilde{\psi}_{t_n,\tilde{\psi}(t_n)}(t_{n+1}) - \tilde{\psi}_{t_n,\phi^n}(t_{n+1}) = \hat{S}_{\Delta t}(\tilde{\psi}(t_n) - \phi^n) + Z.$$

And under Hypothesis 10, we have

$$\mathbf{E}|\tilde{\psi}_{t_n,\tilde{\psi}(t_n)}(t_{n+1}) - \tilde{\psi}_{t_n,\phi^n}(t_{n+1})|_\alpha^2 \leq \mathbf{E}|\tilde{\psi}(t_n) - \phi^n|_\alpha^2(1 + K\Delta t),$$
$$\mathbf{E}|Z|_\alpha^2 \leq K\Delta t\mathbf{E}|\tilde{\psi}(t_n) - \phi^n|_\alpha^2.$$

Proof. Since for $\phi = \tilde{\psi}(t_n)$ or ϕ^n: $\tilde{\psi}_{t_n,\phi}(t_{n+1}) = \hat{S}_{\Delta t}\phi + \Upsilon(\tilde{\psi}_{t_n,\phi}(r))$, we have

$$\tilde{\psi}_{t_n,\tilde{\psi}(t_n)}(t_{n+1}) - \tilde{\psi}_{t_n,\phi^n}(t_{n+1}) = \hat{S}_{\Delta t}(\tilde{\psi}(t_n) - \phi^n) + Z,$$

where

$$Z = \Upsilon(\tilde{\psi}_{t_n,\tilde{\psi}(t_n)}(r)) - \Upsilon(\tilde{\psi}_{t_n,\phi^n}(r)).$$

Then

$$(23) \quad \mathbf{E}|\tilde{\psi}_{t_n,\tilde{\psi}(t_n)}(t_{n+1}) - \tilde{\psi}_{t_n,\phi^n}(t_{n+1})|_\alpha^2$$
$$= \mathbf{E}|\hat{S}_{\Delta t}(\tilde{\psi}(t_n) - \phi^n)|_\alpha^2 + \mathbf{E}|Z|_\alpha^2 + 2\mathbf{E}\langle\hat{S}_{\Delta t}(\tilde{\psi}(t_n) - \phi^n), Z\rangle_\alpha$$
$$= \mathbf{E}|\hat{S}_{\Delta t}(\tilde{\psi}(t_n) - \phi^n)|_\alpha^2 + \mathbf{E}|Z|_\alpha^2 + 2\mathbf{E}\langle\hat{S}_{\Delta t}(\tilde{\psi}(t_n) - \phi^n), \mathbf{E}^{t_n}(Z)\rangle_\alpha,$$

where $\mathbf{E}^{t_n}(\cdot)$ denotes the conditional expectation with respect to $\mathcal{F}_{t_n}$. By Hypothesis 10 and $|S(t)|_{\mathcal{L}(U_c,U_c)} \leq 1$, we have

$$\mathbf{E}|Z|_\alpha^2 \leq 2\mathbf{E}\Big|\int_{t_n}^{t_{n+1}} S(t_{n+1}-r)\Big(F(\tilde{\psi}_{t_n,\tilde{\psi}(t_n)}(r)) - F(\tilde{\psi}_{t_n,\phi^n}(r))\Big)dr\Big|_\alpha^2$$
$$+ 2\mathbf{E}\Big|\int_{t_n}^{t_{n+1}} S(t_{n+1}-r)\Big(G(\tilde{\psi}_{t_n,\tilde{\psi}(t_n)}(r)) - G(\tilde{\psi}_{t_n,\phi^n}(r))\Big)dW(r)\Big|_\alpha^2$$
$$\leq K\int_{t_n}^{t_{n+1}} \mathbf{E}|\tilde{\psi}_{t_n,\tilde{\psi}(t_n)}(r) - \tilde{\psi}_{t_n,\phi^n}(r)|_\alpha^2 dr.$$

For the third term on the right-hand side of (23), by Young's inequality we have

$$\mathbf{E}\langle \hat{S}_{\Delta t}(\tilde{\psi}(t_n) - \phi^n), \mathbf{E}^{t_n}(Z)\rangle_\alpha \leq K\Delta t \mathbf{E}|\tilde{\psi}(t_n) - \phi^n|_\alpha^2 + K\Delta t^{-1}\mathbf{E}|\mathbf{E}^{t_n}(Z)|_\alpha^2$$

$$\leq K\Delta t\mathbf{E}|\tilde{\psi}(t_n) - \phi^n|_\alpha^2 + K\int_{t_n}^{t_{n+1}} \mathbf{E}|\tilde{\psi}_{t_n,\tilde{\psi}(t_n)}(r) - \tilde{\psi}_{t_n,\phi^n}(r)|_\alpha^2 dr.$$

Combining these into the right-hand side of (23) and via Gronwall's lemma, the proposition is proved. □

We now propose the mean-square convergence theorem for numerical method (21).

THEOREM 14. *Assume that Hypotheses* 10, 11, *and* 12 *hold, and there exists a positive constant p such that the method* (21) *satisfies the following conditions:*

$$(24) \qquad \left(\mathbf{E}|\mathbf{E}^{t_n}(\Upsilon(\tilde{\psi}_{t_n,\phi^n}(r)) - \Gamma(\phi^n, \phi^{n+1}, \Delta t, \Delta W_n, T_{\Delta t}))|_\alpha^2\right)^{\frac{1}{2}} \leq K\Delta t^{p+1},$$

$$(25) \qquad \left(\mathbf{E}|\Upsilon(\tilde{\psi}_{t_n,\phi^n}(r)) - \Gamma(\phi^n, \phi^{n+1}, \Delta t, \Delta W_n, T_{\Delta t})|_\alpha^2\right)^{\frac{1}{2}} \leq K\Delta t^{p+\frac{1}{2}}.$$

Then for $n = 1, 2, \ldots, N$, *we have*

$$\left(\mathbf{E}|\psi(t_n) - \phi^n|_\alpha^2\right)^{1/2} \leq K\Delta t^{\min\{q,p\}}.$$

Before the proof of Theorem 14, we first present a lemma which deals with the mean-square bound of numerical solution under the conditions (24)–(25).

LEMMA 15. *Under Hypothesis* 10, *for all natural number N and all* $n = 0, 1, \ldots, N$, *the following inequality holds:*

$$\mathbf{E}|\phi^n|_\alpha^2 \leq K(1 + \mathbf{E}|\varphi|_\alpha^2) \leq K.$$

Proof. From the definition of $\tilde{\psi}_{t_n,\phi^n}(t_{n+1})$: $\tilde{\psi}_{t_n,\phi^n}(t_{n+1}) = \hat{S}_{\Delta t}\phi^n + \Upsilon(\tilde{\psi}_{t_n,\phi^n}(r))$, one obtains easily

$$\mathbf{E}|\tilde{\psi}_{t_n,\phi^n}(t_{n+1})|_\alpha^2 \leq \mathbf{E}|\phi^n|_\alpha^2 + 2\mathbf{E}\Big|\int_{t_n}^{t_{n+1}} S(t_{n+1} - r)F(\tilde{\psi}_{t_n,\phi^n}(r))dr\Big|_\alpha^2$$

$$+ 2\mathbf{E}\Big|\int_{t_n}^{t_{n+1}} S(t_{n+1} - r)G(\tilde{\psi}_{t_n,\phi^n}(r))dW(r)\Big|_\alpha^2$$

$$+ 2\mathbf{E}\langle\hat{S}_{\Delta t}\phi^n, \int_{t_n}^{t_{n+1}} S(t_{n+1} - r)F(\tilde{\psi}_{t_n,\phi^n}(r))dr\rangle_\alpha$$

$$+ 2\mathbf{E}\langle\hat{S}_{\Delta t}\phi^n, \int_{t_n}^{t_{n+1}} S(t_{n+1} - r)G(\tilde{\psi}_{t_n,\phi^n}(r))dW(r)\rangle_\alpha.$$

Since

$$\mathbf{E}\langle\hat{S}_{\Delta t}\phi^n, \int_{t_n}^{t_{n+1}} S(t_{n+1} - r)G(\tilde{\psi}_{t_n,\phi^n}(r))dW(r)\rangle_\alpha$$

$$= \mathbf{E}\langle\hat{S}_{\Delta t}\phi^n, \mathbf{E}^{t_n}\int_{t_n}^{t_{n+1}} S(t_{n+1} - r)G(\tilde{\psi}_{t_n,\phi^n}(r))dW(r)\rangle_\alpha = 0$$

and functions F and $G(\cdot)\mathbf{Q}^{\frac{1}{2}}$ are global Lipchitz which lead to the linear growth property, we have

$$\mathbf{E}|\tilde{\psi}_{t_n,\phi^n}(t_{n+1})|_\alpha^2 \le (1 + K\Delta t)\mathbf{E}|\phi^n|_\alpha^2 + K\mathbf{E}\int_{t_n}^{t_{n+1}} |\tilde{\psi}_{t_n,\phi^n}(r)|_\alpha^2 dr.$$

Then Gronwall's inequality leads to $\mathbf{E}|\tilde{\psi}_{t_n,\phi^n}(t_{n+1})|_\alpha^2 \le (1 + K\Delta t)\mathbf{E}|\phi^n|_\alpha^2$. Similarly, one has the estimations

$$\mathbf{E}|\tilde{\psi}_{t_n,\phi^n}(t_{n+1}) - \hat{S}_{\Delta t}\phi^n|_\alpha^2 \le K\Delta t(1 + \mathbf{E}|\phi^n|_\alpha^2),$$
$$\mathbf{E}|\mathbf{E}^{t_n}(\tilde{\psi}_{t_n,\phi^n}(t_{n+1}) - \hat{S}_{\Delta t}\phi^n)|_\alpha^2 \le K\Delta t^2(1 + \mathbf{E}|\phi^n|_\alpha^2).$$

Suppose that $\mathbf{E}|\phi^n|_\alpha^2 < \infty$; then

$$\mathbf{E}|\phi^{n+1}|_\alpha^2 \le K\mathbf{E}|\tilde{\psi}_{t_n,\phi^n}(t_{n+1})|_\alpha^2 + K\mathbf{E}|\phi^{n+1} - \tilde{\psi}_{t_n,\phi^n}(t_{n+1})|_\alpha^2 < \infty.$$

Since $\mathbf{E}|\varphi|_\alpha^2 < \infty$, we have proved the existence of $\mathbf{E}|\phi^n|_\alpha^2 \ \forall n \in \{0, 1, \dots, N\}$. Thus from

$$\phi^{n+1} = \left(\phi^{n+1} - \tilde{\psi}_{t_n,\phi^n}(t_{n+1})\right) + \left(\tilde{\psi}_{t_n,\phi^n}(t_{n+1}) - \hat{S}_{\Delta t}\phi^n\right) + \hat{S}_{\Delta t}\phi^n,$$

we have

$$\begin{aligned}
\mathbf{E}|\phi^{n+1}|_\alpha^2 &\le \mathbf{E}|\phi^n|_\alpha^2 + 2\mathbf{E}|\Upsilon(\tilde{\psi}_{t_n,\phi^n}(r)) - \Gamma(\phi^n, \phi^{n+1}, \Delta t, \Delta W_n, T_{\Delta t})|_\alpha^2 \\
&\quad + \mathbf{E}\langle\hat{S}_{\Delta t}\phi^n, \mathbf{E}^{t_n}(\Upsilon(\tilde{\psi}_{t_n,\phi^n}(r)) - \Gamma(\phi^n, \phi^{n+1}, \Delta t, \Delta W_n, T_{\Delta t}))\rangle_\alpha \\
&\quad + \mathbf{E}\langle\hat{S}_{\Delta t}\phi^n, \mathbf{E}^{t_n}(\tilde{\psi}_{t_n,\phi^n}(t_{n+1}) - \hat{S}_{\Delta t}\phi^n)\rangle_\alpha + 2\mathbf{E}|\tilde{\psi}_{t_n,\phi^n}(t_{n+1}) - \hat{S}_{\Delta t}\phi^n|_\alpha^2 \\
&\le (1 + K\Delta t)\mathbf{E}|\phi^n|_\alpha^2 + K\Delta t.
\end{aligned}$$

Hence the proof is finished by induction. □

Now we are in position for the proof of Theorem 14.

Proof. First we consider the mean-square error between $\tilde{\psi}(t_{n+1})$ and ϕ^{n+1}. Since

$$\begin{aligned}
\tilde{\psi}(t_{n+1}) - \phi^{n+1} &= \tilde{\psi}_{t_n,\tilde{\psi}(t_n)}(t_{n+1}) - \phi^{n+1} \\
&= \left(\tilde{\psi}_{t_n,\tilde{\psi}(t_n)}(t_{n+1}) - \tilde{\psi}_{t_n,\phi^n}(t_{n+1})\right) + \left(\tilde{\psi}_{t_n,\phi^n}(t_{n+1}) - \phi^{n+1}\right) \\
&= \left(\tilde{\psi}_{t_n,\tilde{\psi}(t_n)}(t_{n+1}) - \tilde{\psi}_{t_n,\phi^n}(t_{n+1})\right) + \left(\Upsilon(\tilde{\psi}_{t_n,\phi^n}(r)) - \Gamma(\phi^n, \phi^{n+1}, \Delta t, \Delta W_n, T_{\Delta t})\right),
\end{aligned}$$

then

$$\begin{aligned}
&\mathbf{E}|\tilde{\psi}(t_{n+1}) - \phi^{n+1}|_\alpha^2 \\
&= \mathbf{E}|\tilde{\psi}_{t_n,\tilde{\psi}(t_n)}(t_{n+1}) - \tilde{\psi}_{t_n,\phi^n}(t_{n+1})|_\alpha^2 + \mathbf{E}|\Upsilon(\tilde{\psi}_{t_n,\phi^n}(r)) - \Gamma(\phi^n, \phi^{n+1}, \Delta t, \Delta W_n, T_{\Delta t})|_\alpha^2 \\
&\quad + 2\mathbf{E}\langle\tilde{\psi}_{t_n,\tilde{\psi}(t_n)}(t_{n+1}) - \tilde{\psi}_{t_n,\phi^n}(t_{n+1}), \Upsilon(\tilde{\psi}_{t_n,\phi^n}(r)) - \Gamma(\phi^n, \phi^{n+1}, \Delta t, \Delta W_n, T_{\Delta t})\rangle_\alpha.
\end{aligned}$$

From Proposition 13, we have

$$\mathbf{E}|\tilde{\psi}_{t_n,\tilde{\psi}(t_n)}(t_{n+1}) - \tilde{\psi}_{t_n,\phi^n}(t_{n+1})|_\alpha^2 \le (1 + K\Delta t)\mathbf{E}|\tilde{\psi}(t_n) - \phi^n|_\alpha^2,$$

and from condition (25), we have

$$\mathbf{E}|\Upsilon(\tilde{\psi}_{t_n,\phi^n}(r)) - \Gamma(\phi^n, \phi^{n+1}, h, \Delta W_n, T_{\Delta t})|_\alpha^2 \le K\Delta t^{2p+1}.$$

From Proposition 13, we may split $\tilde{\psi}_{t_n,\tilde{\psi}(t_n)}(t_{n+1}) - \tilde{\psi}_{t_n,\phi^n}(t_{n+1})$ as $\hat{S}_{\Delta t}(\tilde{\psi}(t_n) - \phi^n)$ and Z, and use the trick of conditional expectation with respect to $\mathcal{F}_{t_n}$, then get

$$\mathbf{E}\langle\tilde{\psi}_{t_n,\tilde{\psi}(t_n)}(t_{n+1}) - \tilde{\psi}_{t_n,\phi^n}(t_{n+1}), \Upsilon(\tilde{\psi}_{t_n,\phi^n}(r)) - \Gamma(\phi^n,\phi^{n+1},\Delta t,\Delta W_n,T_{\Delta t})\rangle_\alpha$$
$$\leq K\Delta t\mathbf{E}|\tilde{\psi}(t_n) - \phi^n|_\alpha^2 + K\Delta t^{2p+1}.$$

Therefore $\mathbf{E}|\tilde{\psi}(t_{n+1}) - \phi^{n+1}|_\alpha^2 \leq (1 + K\Delta t)\mathbf{E}|\tilde{\psi}(t_n) - \phi^n|_\alpha^2 + K\Delta t^{2p+1}$. Hence by Gronwall's lemma, we obtain $\mathbf{E}|\tilde{\psi}(t_{n+1}) - \phi^{n+1}|_\alpha^2 \leq K\Delta t^{2p}$.

Next we consider the estimate between $\psi(t_{n+1})$ and $\tilde{\psi}(t_{n+1})$, since

$$\psi(t_{n+1}) = S(t_{n+1})\varphi + \mathbf{i}\sum_{k=0}^{n}\int_{t_k}^{t_{k+1}} S(t_{n+1} - r)F(\psi(r))dr$$
$$- \mathbf{i}\sum_{k=0}^{n}\int_{t_k}^{t_{k+1}} S(t_{n+1} - r)G(\psi(r))dW(r),$$

and using (22) recurrently

$$\tilde{\psi}(t_{n+1}) = \hat{S}_{\Delta t}^{n+1}\varphi + \mathbf{i}\sum_{k=0}^{n}\int_{t_k}^{t_{k+1}} \hat{S}_{\Delta t}^{n-k}S(t_{k+1} - r)F(\tilde{\psi}(r))dr$$
$$- \mathbf{i}\sum_{k=0}^{n}\int_{t_k}^{t_{k+1}} \hat{S}_{\Delta t}^{n-k}S(t_{k+1} - r)G(\tilde{\psi}(r))dW(r),$$

then

$$\psi(t_{n+1}) - \tilde{\psi}(t_{n+1}) = (S(t_{n+1}) - \hat{S}_{\Delta t}^{n+1})\varphi$$
$$+ \mathbf{i}\sum_{k=0}^{n}\int_{t_k}^{t_{k+1}} \Big(S(t_{n+1} - r)F(\psi(r)) - \hat{S}_{\Delta t}^{n-k}S(t_{k+1} - r)F(\tilde{\psi}(r))\Big)dr$$
$$- \mathbf{i}\sum_{k=0}^{n}\int_{t_k}^{t_{k+1}} \Big(S(t_{n+1} - r)G(\psi(r)) - \hat{S}_{\Delta t}^{n-k}S(t_{k+1} - r)G(\tilde{\psi}(r))\Big)dW(r)$$
$$:= \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3.$$

The estimation of $\mathcal{T}_1$ follows from Hypothesis 12, that is $|\mathcal{T}_1|_\alpha \leq |S(t_{n+1}) - \hat{S}_{\Delta t}^{n+1}|_{\mathcal{L}(\mathbb{H}^\beta,\mathbb{H}^\alpha)}|\varphi|_\beta \leq K\Delta t^q$. By the Lipschitz property of F, we have

$$\mathbf{E}|\mathcal{T}_2|_\alpha^2 \leq K\sum_{k=0}^{n}\int_{t_k}^{t_{k+1}} \mathbf{E}\Big|S(t_{n+1} - r)F(\psi(r)) - \hat{S}_{\Delta t}^{n-k}S(t_{k+1} - r)F(\tilde{\psi}(r))\Big|_\alpha^2 dr$$
$$\leq K\sum_{k=0}^{n}\int_{t_k}^{t_{k+1}} \mathbf{E}\Big|S(t_{n+1} - r)\big(F(\psi(r)) - F(\tilde{\psi}(r))\big)\Big|_\alpha^2 dr$$
$$+ K\sum_{k=0}^{n}\int_{t_k}^{t_{k+1}} \mathbf{E}\Big|\big(S(t_{n+1} - t_{k+1}) - \hat{S}_{\Delta t}^{n-k}\big)S(t_{k+1} - r)F(\tilde{\psi}(r))\Big|_\alpha^2 dr$$

$$\leq K \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} \mathbf{E} \Big| \psi(r) - \tilde{\psi}(r) \Big|_{\alpha}^2 dr + K \Delta t^{2q} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} \mathbf{E} \Big| F(\tilde{\psi}(r)) \Big|_{\beta}^2 dr$$

$$\leq K \Delta t^{2q} + K \int_{t_0}^{t_{n+1}} \mathbf{E} \Big| \psi(r) - \tilde{\psi}(r) \Big|_{\alpha}^2 dr.$$

For term $\mathcal{T}_3$, we use property (3) of the stochastic integral to get

$$\mathbf{E}|\mathcal{T}_3|_{\alpha}^2 = \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} \mathbf{E} \Big| \Big(S(t_{n+1}-r) G(\psi(r)) - \hat{S}_{\Delta t}^{n-k} S(t_{k+1}-r) G(\tilde{\psi}(r)) \Big) \mathbf{Q}^{\frac{1}{2}} \Big|_{HS(U, \mathbb{H}^\alpha)}^2 dr$$

$$\leq K \Delta t^{2q} + K \int_{t_0}^{t_{n+1}} \mathbf{E} \Big| \psi(r) - \tilde{\psi}(r) \Big|_{\alpha}^2 dr.$$

Hence combining them together and by Gronwall's lemma, we have

$$\mathbf{E}|\psi(t_{n+1}) - \tilde{\psi}(t_{n+1})|_{\alpha}^2 \leq K \Delta t^{2q}.$$

At last, we estimate the mean-square error between $\psi(t_{n+1})$ and ϕ^{n+1} by the triangle inequality,

$$\mathbf{E}|\psi(t_{n+1}) - \phi^{n+1}|_{\alpha}^2 \leq 2\mathbf{E}|\psi(t_{n+1}) - \tilde{\psi}(t_{n+1})|_{\alpha}^2 + 2\mathbf{E}|\tilde{\psi}(t_{n+1}) - \phi^{n+1}|_{\alpha}^2$$
$$\leq K \Delta t^{2q} + K \Delta t^{2p} \leq K \Delta t^{\min\{2q, 2p\}}.$$

The proof of the convergence theorem is completed. $\square$

Remark 16. Consider the stochastic Schrödinger equation in the Stratonovich sense,

$$\mathbf{i} d\psi + (A\psi + F(\psi)) dt = G(\psi) \circ dW.$$

It is well known that this equation is equivalent to the following equation in the sense of Itô:

$$\mathbf{i} d\psi + (A\psi + \tilde{F}(\psi)) dt = G(\psi) dW,$$

where $\aleph_{\mathbf{Q}} = \sum_{i \in \mathbb{N}} (\mathbf{Q}^{\frac{1}{2}} e_i(x))^2$ and $\tilde{F}(\psi) = F(\psi) + \frac{\mathbf{i}}{2} G'(\psi) G(\psi) \aleph_{\mathbf{Q}}$. We assume that the coefficients $\tilde{F}$ and G of the equation satisfy Hypothesis 10 and 11; then it is not difficult to understand that Theorem 14 remains true for the equation understood in the sense of Stratonovich.

4.1. Mean-square convergence order of the midpoint scheme. Here we use the convergence theorem (Theorem 14) to obtain the mean-square convergence order of the symplectic semidiscrete scheme (18) with F satisfying the following condition: F is twice Fréchet differentiable with bounded derivatives. Otherwise, truncation strategy could be employed as in [7, 17].

In order to obtain the mean-square convergence order, we require three extra regularities of the solution for (1), namely, we set $\beta = \alpha + 3$. In this case, we have the following estimations to operators (19) [7], which are useful in the calculation below:

$$(26) \quad |\hat{S}_{\Delta t}|_{\mathcal{L}(U_c, U_c)} \leq 1, \qquad |T_{\Delta t}|_{\mathcal{L}(U_c, U_c)} \leq 1, \qquad |S(t_n) - \hat{S}_{\Delta t}^n|_{\mathcal{L}(\mathbb{H}^{\alpha+3}, \mathbb{H}^\alpha)} \leq K \Delta t,$$
$$|\hat{S}_{\Delta t} - I|_{\mathcal{L}(\mathbb{H}^{\alpha+2}, \mathbb{H}^\alpha)} \leq K \Delta t, \qquad |T_{\Delta t} - I|_{\mathcal{L}(\mathbb{H}^{\alpha+2}, \mathbb{H}^\alpha)} \leq K \Delta t.$$

The last inequality in the first line of (26) means that $q = 1$.

Let ξ_i, $i \in \mathbb{N}$ be $\mathcal{N}(0,1)$-distributed random variable. Due to the implicity in diffusion and the possibility of the noise could become unbounded for any arbitrary small time step size, as in [19] we truncate the noise $\Delta W_n = \sum_{i\in\mathbb{N}} \Delta\beta_i \mathbf{Q}^{\frac{1}{2}} e_i = \sqrt{\Delta t}\sum_{i\in\mathbb{N}} \xi_i \mathbf{Q}^{\frac{1}{2}} e_i$ in (18) by another random variable,

$$\Delta\bar{W}_n = \sqrt{\Delta t}\sum_{i\in\mathbb{N}} \zeta_i \mathbf{Q}^{\frac{1}{2}} e_i.$$

For $A_{\Delta t} = \sqrt{2k|\ln\Delta t|}$ $(k \geq 1)$, let

$$\zeta_i = \begin{cases} \xi_i & |\xi_i| \leq A_{\Delta t}, \\ A_{\Delta t} & \xi_i > A_{\Delta t}, \\ -A_{\Delta t} & \xi_i < -A_{\Delta t}. \end{cases}$$

Here, the role of parameter k is such that $\mathbf{E}(\zeta_i - \xi_i)^2 \leq \Delta t^k$ and $\mathbf{E}(\zeta_i^2 - \xi_i^2) \leq (1 + 2A_{\Delta t})\Delta t^k$. Thus for $\mathbf{Q}^{\frac{1}{2}} \in HS(U, \mathbb{H}^\alpha)$, we have

$$(27) \qquad \begin{aligned} &\mathbf{E}|\Delta\bar{W}_n - \Delta W_n|_\alpha^2 \leq K\Delta t^{k+1}, \\ &\mathbf{E}|(\Delta\bar{W}_n)^2 - (\Delta W_n)^2|_\alpha^2 \leq K\Delta t^{k+2}(1 + A_{\Delta t} + A_{\Delta t}^2) \leq K\Delta t^{k+1}, \end{aligned}$$

where we use the property $\Delta t(1 + A_{\Delta t} + (A_{\Delta t})^2) \leq 1$ for sufficient small Δt.

For the following analysis, we require $k = 2$ and have the following results.

THEOREM 17. *The midpoint scheme is of order 1 in the mean-square convergence sense, i.e., for $\alpha > \frac{d}{2}$ with d being the dimension of the problem and for sufficient small Δt, we have*

$$\left(\mathbf{E}|\psi(t_n) - \phi^n|_\alpha^2\right)^{\frac{1}{2}} \leq K\Delta t \qquad \forall n = 0, 1, \ldots, N.$$

Proof. First of all, we show that the numerical solution (18) with truncated noise has bounded moments. In fact,

$$\begin{aligned} (28) \quad \mathbf{E}|\phi^{n+1}|_\alpha^2 = &\mathbf{E}\Big| \hat{S}_{\Delta t}^{n+1}\varphi + \mathbf{i}\Delta t\sum_{k=0}^n \hat{S}_{\Delta t}^{n-k} T_{\Delta t} F(t_{k+\frac{1}{2}}, \phi^{k+\frac{1}{2}}) \\ &- \mathbf{i}\varepsilon\sum_{k=0}^n \hat{S}_{\Delta t}^{n-k} T_{\Delta t}\Big(\frac{1}{2}(I + \hat{S}_{\Delta t})\phi^k + \frac{1}{2}(\phi^{k+1} - \hat{S}_{\Delta t}\phi^k)\Big)\Delta\bar{W}_k\Big|_\alpha^2 \\ \leq &K\mathbf{E}|\varphi|_\alpha^2 + K\Delta t\sum_{k=0}^n \mathbf{E}|\phi^{k+\frac{1}{2}}|_\alpha^2 + Kn\sum_{k=0}^n \big|(\phi^{k+1} - \hat{S}_{\Delta t}\phi^k)\Delta\bar{W}_k\big|_\alpha^2. \end{aligned}$$

Since

$$\begin{aligned} &\mathbf{E}\big|(\phi^{k+1} - \hat{S}_{\Delta t}\phi^k)\Delta\bar{W}_k\big|_\alpha^2 \\ &= \mathbf{E}\Big|\mathbf{i}\Delta t T_{\Delta t} F(t_{k+\frac{1}{2}}, \phi^{k+\frac{1}{2}})\Delta\bar{W}_k \\ &\quad - \mathbf{i}\varepsilon T_{\Delta t}\phi^k(\Delta\bar{W}_k)^2 - \frac{\mathbf{i}}{2}\varepsilon T_{\Delta t}\big((\phi^{k+1} - \hat{S}_{\Delta t}\phi^k)\Delta\bar{W}_k\big)\Delta\bar{W}_k\Big|_\alpha^2 \\ &\leq K\Delta t^2\Delta t(A_{\Delta t})^2\mathbf{E}|\phi^{k+\frac{1}{2}}|_\alpha^2 + K\Delta t^2\mathbf{E}|\phi^k|_\alpha^2 + K_0\Delta t(A_{\Delta t})^2\mathbf{E}\big|(\phi^{k+1} - \hat{S}_{\Delta t}\phi^k)\Delta\bar{W}_k\big|_\alpha^2. \end{aligned}$$

There exists Δt^* such that $\forall\, \Delta t < \Delta t^*$ one has $K_0 \Delta t(A_{\Delta t})^2 \leq \frac{1}{2}$; therefore

$$(29) \qquad \mathbf{E}\big|(\phi^{k+1} - \hat{S}_{\Delta t}\phi^k)\Delta\bar{W}_k\big|_\alpha^2 \leq K\Delta t^2 \mathbf{E}|\phi^{k+\frac{1}{2}}|_\alpha^2.$$

Substituting (29) into the right-hand side of (28) and by the Gronwall inequality one arrives at

$$\mathbf{E}|\phi^n|_\alpha^2 \leq K(1 + \mathbf{E}|\varphi|_\alpha^2) \leq K.$$

From (5), we know that

$$\Upsilon(\tilde{\psi}_{t_n,\phi^n}(r)) = \mathbf{i}\int_{t_n}^{t_{n+1}} S(t_{n+1}-r)F(r,\tilde{\psi}_{t_n,\phi^n}(r))dr$$
$$- \mathbf{i}\varepsilon \int_{t_n}^{t_{n+1}} S(t_{n+1}-r)\tilde{\psi}_{t_n,\phi^n}(r)dW(r)$$
$$- \frac{\varepsilon^2}{2}\int_{t_n}^{t_{n+1}} S(t_{n+1}-r)\tilde{\psi}_{t_n,\phi^n}(r)\aleph_{\mathbf{Q}}dr,$$

and from (18) with truncated noise we have

$$\Gamma(\phi^n,\phi^{n+1},\Delta t,\Delta\bar{W}_n,T_{\Delta t}) = \mathbf{i}\Delta t T_{\Delta t}F(t_{n+\frac{1}{2}},\phi^{n+\frac{1}{2}})$$
$$- \mathbf{i}\varepsilon T_{\Delta t}\phi^n\Delta\bar{W}_n - \frac{\mathbf{i}\varepsilon}{2}T_{\Delta t}(\phi^{n+1} - \phi^n)\Delta\bar{W}_n.$$

Then

$$\mathbf{E}^{t_n}\Big(\Upsilon(\tilde{\psi}_{t_n,\phi^n}(r)) - \Gamma(\phi^n,\phi^{n+1},\Delta t,\Delta\bar{W}_n,T_{\Delta t})\Big)$$
$$= \mathbf{E}^{t_n}\Big(\mathbf{i}\int_{t_n}^{t_{n+1}} S(t_{n+1}-r)F(r,\tilde{\psi}_{t_n,\phi^n}(r))dr - \mathbf{i}\Delta t T_{\Delta t}F(t_{n+\frac{1}{2}},\phi^{n+\frac{1}{2}})\Big)$$
$$+ \mathbf{E}^{t_n}\Big(\frac{\mathbf{i}\varepsilon}{2}T_{\Delta t}(\phi^{n+1} - \phi^n)\Delta\bar{W}_n - \frac{\varepsilon^2}{2}\int_{t_n}^{t_{n+1}} S(t_{n+1}-r)\tilde{\psi}_{t_n,\phi^n}(r)\aleph_{\mathbf{Q}}dr\Big)$$
$$:= \mathcal{A} + \mathcal{B}.$$

We split $\mathcal{A}$ further,

$$(30) \qquad \mathcal{A} = \mathbf{i}\mathbf{E}^{t_n}\int_{t_n}^{t_{n+1}}(S(t_{n+1}-r) - T_{\Delta t})F(r,\tilde{\psi}_{t_n,\phi^n}(r))dr$$
$$+ \mathbf{i}\mathbf{E}^{t_n}\int_{t_n}^{t_{n+1}} T_{\Delta t}(F(r,\tilde{\psi}_{t_n,\phi^n}(r)) - F(t_{n+\frac{1}{2}},\phi^n))dr$$
$$- \mathbf{i}\Delta t T_{\Delta t}\mathbf{E}^{t_n}(F(t_{n+\frac{1}{2}},\phi^{n+\frac{1}{2}}) - F(t_{n+\frac{1}{2}},\phi^n))$$
$$:= \mathcal{A}^1 + \mathcal{A}^2 + \mathcal{A}^3.$$

From (26), we have $\mathbf{E}|\mathcal{A}^1|_\alpha^2 \leq K\Delta t^4$. In addition, via the boundedness of derivatives of F with respect to ψ, we write

$$F(r,\tilde{\psi}_{t_n,\phi^n}(r)) - F(t_{n+\frac{1}{2}},\phi^n) = \Big(F(r,\tilde{\psi}_{t_n,\phi^n}(r)) - F(r,\phi^n)\Big) + \Big(F(r,\phi^n) - F(t_{n+\frac{1}{2}},\phi^n)\Big)$$
$$= \frac{\partial F}{\partial\psi}(r,\phi^n)(\tilde{\psi}_{t_n,\phi^n}(r) - \phi^n) + \rho_1 + \rho_2,$$

where $|\rho_1|_\alpha \le K|\tilde{\psi}_{t_n,\phi^n}(r) - \phi^n|_\alpha^2$ and $\mathbf{E}|\rho_2|_\alpha^2 \le K\Delta t^2$. It is not difficult to obtain the estimate

$$\int_{t_n}^{t_{n+1}} \mathbf{E}|\mathbf{E}^{t_n}(\tilde{\psi}_{t_n,\phi^n}(r) - \phi^n)|_\alpha^2 dr \le K\Delta t^3.$$

The above implies that $\mathbf{E}|\mathcal{A}^2|_\alpha^2 \le K\Delta t^4$. Similarly as the estimate of $\mathcal{A}^2$, one has $\mathbf{E}|\mathcal{A}^3|_\alpha^2 \le K\Delta t^4$. The estimate of $\mathcal{B}$ is more technical. First inserting the expression of $\phi^{n+1} - \phi^n$ into $\mathcal{B}$, we have

$$\begin{aligned}
\mathcal{B} =& \mathbf{E}^{t_n}\Big[\frac{\mathbf{i}\varepsilon}{2}T_{\Delta t}\Big((\hat{S}_{\Delta t} - I)\phi^n + \mathbf{i}\Delta t T_{\Delta t}F(t_{n+\frac{1}{2}}, \phi^{n+\frac{1}{2}}) - \mathbf{i}\varepsilon T_{\Delta t}\phi^n \Delta\bar{W}_n \\
& - \frac{\mathbf{i}\varepsilon}{2}T_{\Delta t}(\phi^{n+1} - \phi^n)\Delta\bar{W}_n\Big)\Delta\bar{W}_n - \frac{\varepsilon^2}{2}\int_{t_n}^{t_{n+1}} S(t_{n+1} - r)\tilde{\psi}_{t_n,\phi^n}(r)\aleph_{\mathbf{Q}}dr\Big] \\
=& -\frac{\varepsilon}{2}\Delta t T_{\Delta t}^2\mathbf{E}^{t_n}\Big(F(t_{n+\frac{1}{2}}, \phi^{n+\frac{1}{2}})\Delta\bar{W}_n\Big) + \frac{\varepsilon^2}{4}T_{\Delta t}^2\mathbf{E}^{t_n}\Big((\phi^{n+1} - \phi^n)(\Delta\bar{W}_n)^2)\Big) \\
& + \frac{\varepsilon^2}{2}\Big[T_{\Delta t}^2\phi^n\mathbf{E}^{t_n}(\Delta\bar{W}_n)^2 - \int_{t_n}^{t_{n+1}} \mathbf{E}^{t_n}S(t_{n+1} - r)\tilde{\psi}_{t_n,\phi^n}(r)\aleph_{\mathbf{Q}}dr\Big] \\
:=& \mathcal{B}^1 + \mathcal{B}^2 + \mathcal{B}^3.
\end{aligned}$$

We write $F(t_{n+\frac{1}{2}}, \phi^{n+\frac{1}{2}}) = F(t_{n+\frac{1}{2}}, \phi^n) + \frac{1}{2}\frac{\partial F}{\partial\psi}(t_{n+\frac{1}{2}}, \phi^n)(\phi^{n+1} - \phi^n) + \rho_1$ and use (29) to obtain $\mathbf{E}|\mathcal{B}^1|_\alpha^2 \le K\Delta t^4$. Inserting the expression of $\phi^{n+1} - \phi^n$ into the term $\mathcal{B}^2$, similarly, one can obtain that $\mathbf{E}|\mathcal{B}^2|_\alpha^2 \le K\Delta t^4$. We split $\mathcal{B}^3$ further,

$$\begin{aligned}
\mathcal{B}^3 =& \frac{\varepsilon^2}{2}T_{\Delta t}\phi^n\mathbf{E}^{t_n}\Big((\Delta\bar{W}_n)^2 - (\Delta W_n)^2\Big) \\
& + \frac{\varepsilon^2}{2}\mathbf{E}^{t_n}\int_{t_n}^{t_{n+1}} \Big(T_{\Delta t}^2\phi^n - S(t_{n+1} - r)\tilde{\psi}_{t_n,\phi^n}(r)\Big)\aleph_{\mathbf{Q}}dr,
\end{aligned}$$

where the first term could be bounded by the second inequality in (27), and the second term in the above equality could be estimated similarly by inserting the expression of $\tilde{\psi}_{t_n,\phi^n}(r) - \phi^n$ into it, which lead to $\mathbf{E}|\mathcal{B}^3|_\alpha^2 \le K\Delta t^4$. Therefore, we have

$$\mathbf{E}\Big|\mathbf{E}^{t_n}\Big(\Upsilon(\tilde{\psi}_{t_n,\phi^n}(r)) - \Gamma(\phi^n, \phi^{n+1}, \Delta t, \Delta\bar{W}_n, T_{\Delta t})\Big)\Big|_\alpha^2 \le K\Delta t^4.$$

Next, let's compute the value of $\mathbf{E}|\Upsilon(\tilde{\psi}_{t_n,\phi^n}(r)) - \Gamma(\phi^n, \phi^{n+1}, \Delta t, \Delta\bar{W}_n, T_{\Delta t})|_\alpha^2$.

$$\begin{aligned}
& \Upsilon(\tilde{\psi}_{t_n,\phi^n}(r)) - \Gamma(\phi^n, \phi^{n+1}, \Delta t, \Delta\bar{W}_n, T_{\Delta t}) \\
=& \mathbf{i}\int_{t_n}^{t_{n+1}} \Big(S(t_{n+1} - r)F(r, \tilde{\psi}_{t_n,\phi^n}(r)) - T_{\Delta t}F(t_{n+\frac{1}{2}}, \phi^{n+\frac{1}{2}})\Big)dr \\
& - \mathbf{i}\varepsilon\int_{t_n}^{t_{n+1}} (S(t_{n+1} - r) - T_{\Delta t})\tilde{\psi}_{t_n,\phi^n}(r)dW(r) \\
& - \Big[\mathbf{i}\varepsilon\int_{t_n}^{t_{n+1}} T_{\Delta t}(\tilde{\psi}_{t_n,\phi^n}(r) - \phi^n)dW(r) + \frac{\varepsilon^2}{2}\int_{t_n}^{t_{n+1}} S(t_{n+1} - r)\tilde{\psi}_{t_n,\phi^n}(r)\aleph_{\mathbf{Q}}dr \\
& - \frac{\mathbf{i}\varepsilon}{2}T_{\Delta t}(\phi^{n+1} - \phi^n)\Delta\bar{W}_n + \mathbf{i}\varepsilon T_{\Delta t}\phi^n\big(\Delta W_n - \Delta\bar{W}_n\big)\Big] \\
:=& \mathcal{L} + \mathcal{M} - \mathcal{N}.
\end{aligned}$$

The estimation of $\mathcal{L}$ is similar to that of term $\mathcal{A}$, that is, $\mathbf{E}|\mathcal{L}|^2_\alpha \leq K\Delta t^3$. By (3) and (26), we know that $\mathbf{E}|\mathcal{M}|^2_\alpha \leq K\Delta t^3$. Inserting the expression of $\tilde{\psi}(r) - \phi^n$ and $\phi^{n+1} - \phi^n$ into $\mathcal{N}$ and splitting $\mathcal{N}$ further,

$$
\begin{aligned}
\mathcal{N} = {} & \mathbf{i}\varepsilon \int_{t_n}^{t_{n+1}} T_{\Delta t}\Big((\hat{S}_{r-t_n} - I)\phi^n + \mathbf{i}\int_{t_n}^r S(r-\rho)\tilde{F}(\tilde{\psi}_{t_n,\phi^n}(\rho))d\rho\Big)dW(r) \\
& - \frac{\mathbf{i}\varepsilon}{2}T_{\Delta t}\Big((\hat{S}_{\Delta t} - I)\phi^n + \mathbf{i}\Delta t T_{\Delta t}F(t_{n+\frac{1}{2}}, \phi^{n+\frac{1}{2}})\Big)\Delta\bar{W}_n + \mathbf{i}\varepsilon T_{\Delta t}\phi^n\big(\Delta W_n - \Delta\bar{W}_n\big) \\
& + \varepsilon^2 \int_{t_n}^{t_{n+1}} \int_{t_n}^r T_{\Delta t}S(r-\rho)\tilde{\psi}_{t_n,\phi^n}(\rho)dW(\rho)dW(r) \\
& + \frac{\varepsilon^2}{2}\int_{t_n}^{t_{n+1}} S(t_{n+1}-r)\tilde{\psi}_{t_n,\phi^n}(r)\aleph_{\mathbf{Q}}dr \\
& - \frac{\varepsilon^2}{2}T^2_{\Delta t}\phi^n(\Delta W_n)^2 + \frac{\varepsilon^2}{2}T^2_{\Delta t}\phi^n\Big((\Delta W_n)^2 - (\Delta\bar{W}_n)^2\Big) - \frac{\varepsilon^2}{4}T^2_{\Delta t}(\phi^{n+1} - \phi^n)(\Delta\bar{W}_n)^2.
\end{aligned}
$$

Special attention should be paid to the third line and the first term in the last line of the above equality, since other terms could be estimated as before. In fact,

$$
\frac{\varepsilon^2}{2}T^2_{\Delta t}\phi^n(\Delta W_n)^2 = \varepsilon^2 \int_{t_n}^{t_{n+1}} \int_{t_n}^r T^2_{\Delta t}\phi^n dW(\rho)dW(r) + \frac{\varepsilon^2}{2}T^2_{\Delta t}\phi^n\aleph_{\mathbf{Q}}\Delta t
$$

leads to $\mathbf{E}|\mathcal{N}|^2_\alpha \leq K\Delta t^3$. That is,

$$
\Big(\mathbf{E}|\Upsilon(\tilde{\psi}_{t_n,\phi^n}(r)) - \Gamma(\phi^n, \phi^{n+1}, \Delta t, \Delta\bar{W}_n, T_{\Delta t})|^2_\alpha\Big)^{\frac{1}{2}} \leq K\Delta t^{\frac{3}{2}}.
$$

Therefore the mean-square order of midpoint scheme is of 1 according to Theorem 14. □

Remark 18. For the stochastic Schrödinger equation in the Stratonovich sense, the mean-square convergence order of the semidiscrete midpoint scheme is 1 under appropriate assumptions. It seems to be the same as the case of stochastic ordinary differential equations. However, the convergence order here depends on the values of p and q, where p and q are from estimations of the one-step deviation between exact solution and numerical solution. As we take parameters $\beta = \alpha + 3$, which require three more regularity conditions on the solution ψ, the estimation of operators is $|S(t_k) - \hat{S}^k_{\Delta t}|_{\mathcal{L}(H^\beta, H^\alpha)} \leq K\Delta t$ with $q = 1$. Together with $p = 1$, we have that the convergence order of the semidiscrete midpoint scheme is 1. If we put less regularity on the solution, which means $q < 1$, then the convergence order is less than 1. For the general stochastic Runge–Kutta methods in the temporal direction, the mean-square convergence order is an open problem.

5. Numerical experiments. In section 4.1, we showed the convergence in the mean-square sense of midpoint scheme (18) with spatially regular noise and under certain assumptions the order is of 1. The following example was chosen to study convergence order computationally of the midpoint scheme (18) to solve the stochastic cubic Schrödinger equation on $[-1, 1] \times [0, T]$:

$$
\begin{aligned}
(31) \qquad & \mathbf{i}d\psi(x,t) + \Big(\partial_{xx}\psi + |\psi|^2\psi(x,t)\Big)dt = \varepsilon\psi(x,t) \circ dW(t), \\
& \psi(x,0) = \sin(\pi x).
\end{aligned}
$$

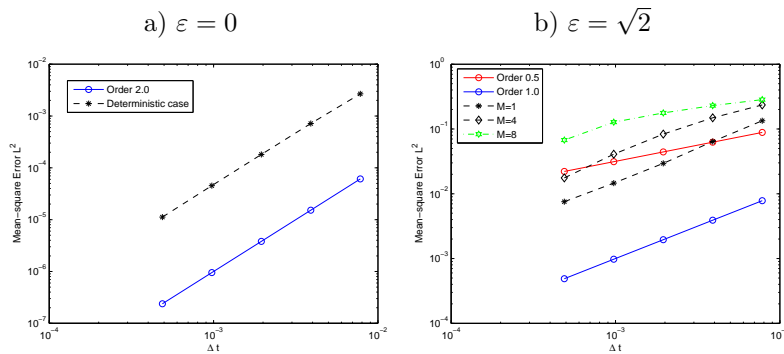a) $\varepsilon = 0$ b) $\varepsilon = \sqrt{2}$

FIG. 1. (a) *Rates of convergence for the deterministic case in the norm* $\|\psi(T) - \phi^N\|_{\mathbb{L}^2}$ *($T = \frac{1}{4}$,* $\varepsilon = 0$, $\Delta x = \frac{1}{256}$, $\Delta t \in \{2^{-i}; 7 \leq i \leq 11\}$*). (b) Rates of convergence for the stochastic nonlinear Schrödinger driven by* $W(t) = \sum_{\ell=1}^{M} \frac{1}{\ell} \sin(\pi\ell x)\beta_\ell(t)$ *in the norm* $\left(E[\|\psi(T) - \phi^N\|_{\mathbb{L}^2}^2]\right)^{1/2}$ *($T = \frac{1}{4}$,* $\varepsilon = \sqrt{2}$, $\Delta t \in \{2^{-i}; 7 \leq i \leq 11\}$*).*
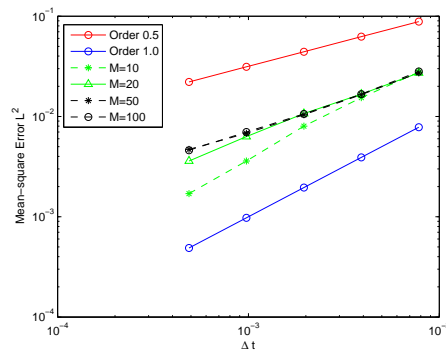

FIG. 2. *Rates of convergence for the stochastic nonlinear Schrödinger driven by* $W(t) = \sum_{\ell=1}^{M} \frac{1}{\ell^2} \sin(\pi\ell x)\beta_\ell(t)$ *in the norm* $\left(E[\|\psi(T) - \phi^N\|_{\mathbb{L}^2}^2]\right)^{1/2}$ *($T = \frac{1}{4}$,* $\varepsilon = \sqrt{2}$, $\Delta t \in \{2^{-i}; 7 \leq i \leq 11\}$*).*

Let $T = \frac{1}{4}$. For integer M, and $\{\beta_\ell; 1 \leq \ell \leq M\}$ a family of independent $\mathbb{R}$-valued Wiener processes, consider the real-valued Wiener process $W(t) = \sum_{\ell=1}^{M} \frac{1}{\ell} \sin(\pi\ell x) \beta_\ell(t)$, and $\varepsilon = \sqrt{2}$ in (31). We use the midpoint scheme (18) in the temporal direction and finite difference in the spatial direction for the numerical approximation. Let $I_{\Delta t} = \{t_n; 0 \leq n \leq N\}$ be the uniform discretization of $[0, T]$ of size $\Delta t > 0$, and apply the uniform discretization of $[-1, 1]$ of size $\Delta x = \frac{1}{256}$. The reference values (for Figure 1(b)) are generated for the smallest mesh size $\Delta t = 2^{-14}$. And 500 realizations are chosen to approximate the expectations.

We consider $\varepsilon = 0$ first: Figure 1(a) shows order 2 for the $\mathbb{L}^2$-error of the midpoint scheme. The observations are different in the stochastic case ($\varepsilon = \sqrt{2}$) where different sorts of Wiener processes depending on M are used: as is displayed in Figure 1(b), the strong order of convergence for $M = 1$ and $M = 4$ is approximately to 1, but it drops to 0.5 approximately for value 8 of M. As stated in Remark 4.6, the convergence order depends on the regularity of the solution which depends on the property of operator $\mathbf{Q}$, so we consider large $M = 50$, the numerical convergence order is only approximate
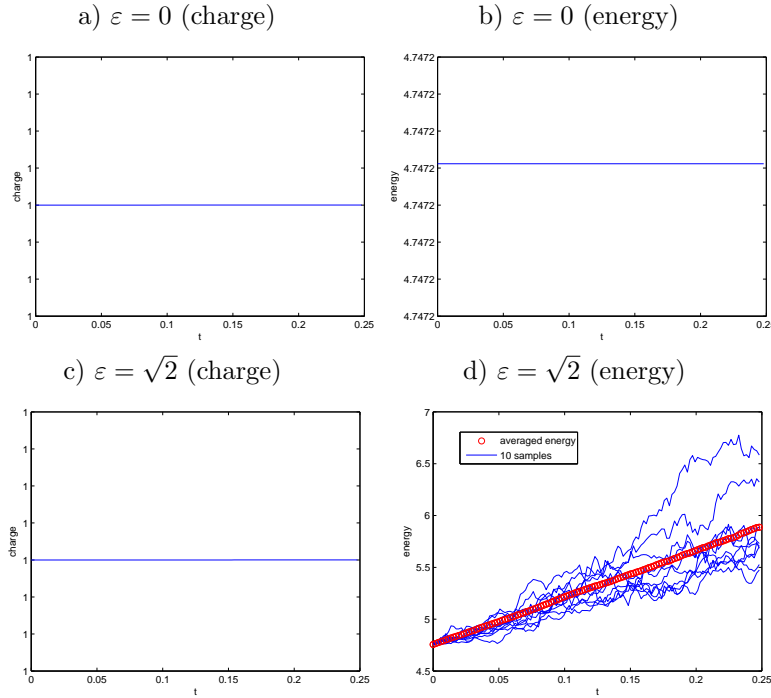
a) $\varepsilon = 0$ (charge)   b) $\varepsilon = 0$ (energy)

c) $\varepsilon = \sqrt{2}$ (charge)   d) $\varepsilon = \sqrt{2}$ (energy)

FIG. 3. (a) *Evolution of charge in deterministic case ($\varepsilon = 0$, $\Delta t = 2^{-9}$, $T = \frac{1}{4}$). (b) Evolution of energy in deterministic case ($\varepsilon = 0$, $\Delta t = 2^{-9}$, $T = \frac{1}{4}$). (c) Evolution of charge in stochastic case ($\varepsilon = \sqrt{2}$, $\Delta t = 2^{-9}$, $T = \frac{1}{4}$). (d) Evolution of energy in stochastic case ($\varepsilon = \sqrt{2}$, $\Delta t = 2^{-9}$, $T = \frac{1}{4}$).*

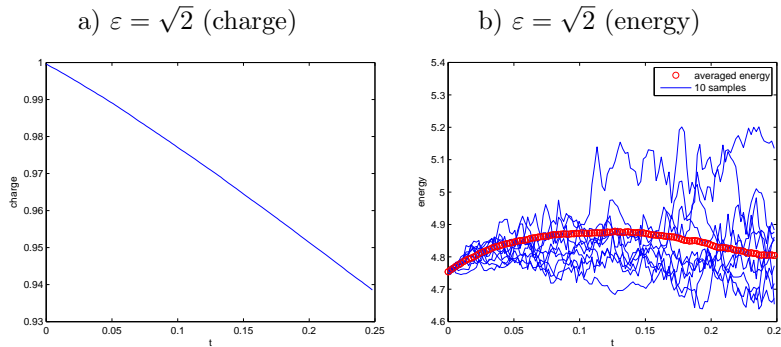a) $\varepsilon = \sqrt{2}$ (charge)   b) $\varepsilon = \sqrt{2}$ (energy)

FIG. 4. (a) *Evolution of charge in stochastic case ($\varepsilon = \sqrt{2}$, $\Delta t = 2^{-9}$, $T = \frac{1}{4}$). (b) Evolution of energy in stochastic case ($\varepsilon = \sqrt{2}$, $\Delta t = 2^{-9}$, $T = \frac{1}{4}$).*

to 0.1. Consider another spatially more smooth real-valued Wiener process $W(t) = \sum_{\ell=1}^{M} \frac{1}{\ell^2} \sin(\pi \ell x) \beta_\ell(t)$; Figure 2 shows that the convergence order is approximately to 1 for $M = 10$, and similarly as before it drops to 0.5 as M grows to 100.

The plots in Figures 3(a) and 3(c) study the conservation of the discrete charge of the of midpoint scheme in both the deterministic and the stochastic case. Figures 3(b) and 3(d) study the evolution of the discrete energy of the midpoint scheme: we may

observe from Figure 3(b) that the discrete energy is preserved by the midpoint scheme in the deterministic case; however, it is no longer a constant in the stochastic case, and we observe a linear growth for the discrete averaged energy over 500 paths.

For comparison with the midpoint scheme (18), we consider a nonstructure preserving numerical method

$$\phi^{n+1} = \phi^n + \mathbf{i}\Delta t A \phi^{n+1} + \mathbf{i}\Delta t F(\phi^{n+1}) - \mathbf{i}\varepsilon \Delta W_n$$

and apply the same spatial discretization. Figure 4 displays the evolution of discrete charge and energy for the above method: the discrete charge is no longer preserved but decreases linearly; the discrete averaged energy does not show linear growth.

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